PRISMPLAY

JONATHAN CHOATE

This month's column deals with volumes of prisms and regular anti-prisms and was inspired by some interesting results I found in Blackwell's *Geometry and Architecture*. I highly recommend this book to anyone who teaches geometry because it contains a wealth of interesting applications of what is currently taught in a plane geometry course. What caught my eye were some results about prisms that could be used as a nice supplement to material that appears in many current geometry textbooks. The results have to do with the total surface area of a prism and its exposed surface area, which is the area of the sides plus the area of the top. The latter is of architectural interest because if you were building a structure in the shape of a prism, the base would not be exposed to the elements. The former makes sense if you were thinking of an interior space such as a room. In what follows, remember that a regular prism is one in which all the edges are equal and all the lateral faces are squares. Here is what Blackwell has to say about which of the regular prisms is the most efficient in the sense that for a given volume the surface area is a minimum.

"Of the regular prisms, the pentagonal prism (not the cube) offers the best volume enclosure when the total surface area is important. The octagonal prism is the most efficient when the exposed surface area, the walls and the roof but not the floor is important."[1]

Verifying this is a fun exercise. First, you have to come up with formula for S, the length of a side of the prism, as a function of the volume V and the number of sides n. Since the area of an n-sided regular polygon with side S is

$$\frac{S^2\cot\left(\frac{180}{n}\right)}{4},$$

the volume V of the prism is given by

$$V = \frac{S^3\cot\left(\frac{180}{n}\right)}{4}.$$

Solving for S gives

$$S = \left(\frac{4V\tan\left(\frac{180}{n}\right)}{n}\right)^{\frac{1}{3}}.$$

The total surface area, TSA, equals twice the area of the base plus n times S^2 and the exposed surface area, ESA, equals the base area plus n times S^2 which gives you the formulas

$$\text{TSA} = \frac{S^2\cot\left(\frac{180}{n}\right)}{2} + nS^2.$$

and

$$\text{ESA} = \frac{S^2\cot\left(\frac{180}{n}\right)}{4} + nS^2.$$

Since we are interested in what happens as we vary n and hold V fixed, a spreadsheet is handy here. The table below was created using Excel and shows what happens to TSA and ESA for a fixed volume of 1000 when you vary the number of sides n in the polygonal bases.

Polygon	Side	Total
3	13.2180125	675.456243
4	9.9999958	599.999735
5	8.3454581	587.884669
6	7.2741551	592.426015
7	6.50441696	603.635443
8	5.91649684	618.077351
9	5.44871782	634.255565
10	5.06537823	651.417814
11	4.74410237	669.147643
12	4.47002828	687.198296
13	4.23283552	705.416163
14	4.02509976	723.702088
15	3.84132415	741.990364
16	3.67734017	760.236652

Polygon	Exposed	Lateral
3	599.801904	524.147565
4	499.99969	399.999646
5	468.059011	348.233354
6	454.953005	317.479994
7	449.893761	296.152079
8	449.058415	280.039479
9	450.726149	267.196733
10	453.99919	256.580566
11	458.359612	247.571581
12	463.486065	239.773834
13	469.167909	232.919655
14	475.26104	226.819993
15	481.663466	221.336569
16	488.300972	216.365292

TABLE 1. TOTAL SURFACE AREA AND EXPOSED SURFACE AREA FOR REGULAR N-GONAL PRISMS WITH A VOLUME OF 1000 CUBIC UNITS.

Just as Blackwell stated, the total surface area, TSA, is a minimum of 587.88 square units when n is 5 and the exposed surface area, ESA, is a minimum of 449.06 square units when n is 8.

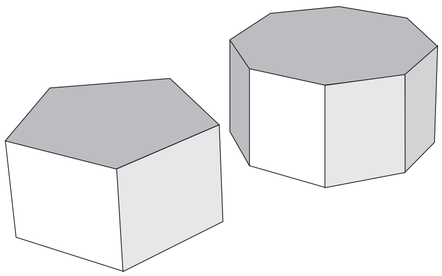

FIGURE 1. TWO REGULAR PRISMS WITH THE SAME VOLUME. A PENTAGONAL REGULAR PRISM WITH MINIMUM TOTAL SURFACE AREA AND AN OCTAGONAL ONE WITH MINIMUM EXPOSED SURFACE AREA

Later in the same chapter, Blackwell states a more general result about prisms:

"In all but the regular prisms, height is a variable. The unique prisms are those in which the height is proportional to the base so as to minimize the surface area.

The shape of the base may be a rectangle of any proportion, a regular or irregular polygon, or any enclosed shape, but when the area of the walls is four times the area of the base, the total surface required to enclose a volume will be the least possible. The shape will be at its most efficient proportion in terms of volume enclosure.

When only the exposed surface is important, the area of the roof and the walls but not the floor, a similar relationship holds. The exposed surface of any prism will be minimal when the area of the walls is twice the area of the floor."[2]

In order to verify this you need to come up with another set of formulas. You need to get S the length of a side of the base as a function of h the height, n the number of sides and the fixed volume. If V is the fixed volume and S is the length of a side, then

$$V = \frac{hnS^3 \cot\left(\frac{180}{n}\right)}{4}.$$

Solving for S yields

$$S = \sqrt[3]{\frac{4V}{hn\cot\left(\frac{180}{n}\right)}}.$$

Using this you get the following formulas for TSA and ESA:

$$\text{TSA} = \frac{S^2\cot\left(\frac{180}{n}\right)}{2} + nhS$$

$$\text{ESA} = \frac{S^2\cot\left(\frac{180}{n}\right)}{4} + nhS.$$

Table 2 is an Excel spreadsheet that uses these formulas with a V of 1000 and an n of 4.

The last two lines show that the exposed surface area approaches a minimum when h is close to 6.3, and the ratio of the area of the base to the area of the walls approaches 2:1 and that the total surface area approaches a minimum when h is close to 10.01 and the ratio of base to walls is close to 4:1.

Volume	#of sides
1000	4

h	base side	Total	Exposed	Walls	Base	Base/Wall
1	31.62275562	2126.49102	1126.49102	126.491022	1000	0.12649102
2	22.36066494	1178.88532	678.88532	178.88532	500	0.35777064
3	18.25740647	885.755544	552.422211	219.088878	333.333333	0.65726663
4	15.81137781	752.982045	502.982045	252.982045	250	1.01192818
5	14.14212624	682.842525	482.842525	282.842525	200	1.41421262
6	12.90993592	643.171795	476.505129	309.838462	166.666667	1.85903077
7	11.95227816	620.378074	477.520931	334.663789	142.857143	2.34264652
8	11.18033247	607.770639	482.770639	357.770639	125	2.86216511
9	10.54091854	601.69529	490.584179	379.473067	111.111111	3.41525761
10	9.999993366	599.999735	499.999735	399.999735	100	3.99999735
11	9.534619567	601.341443	510.432352	419.523261	90.9090909	4.61475587
12	9.128703236	604.844422	521.511089	438.177755	83.3333333	5.25813306
13	8.770574375	609.916021	532.992944	456.069867	76.9230769	5.92890828
14	8.451536941	616.143212	544.71464	473.286069	71.4285714	6.62600496
15	8.164960393	623.230957	556.56429	489.897624	66.6666667	7.34846435
16	7.905688906	630.96409	568.46409	505.96409	62.5	8.09542544
10.01	9.994997116	599.999884	500.099784	400.199685	99.9000999	4.00599884
6.3	12.59880741	634.950264	476.220105	317.489947	158.730159	2.00018666

TABLE 2. SPREADSHEET ANALYSIS FOR FINDING THE HEIGHT OF A SQUARE PRISM WITH A FIXED VOLUME THAT GIVES MINIMUM TSA AND ESA.

Volume	#of sides
1000	8

h	base side	Total	Exposed	Walls	Base	Base/Wall
1	14.3911982	2115.12959	1115.12959	115.129586	1000	0.11512959
2	10.1761139	1162.81782	662.817822	162.817822	500	0.32563564
3	8.30876218	866.076959	532.743626	199.410292	333.333333	0.59823088
4	7.19559912	730.259172	480.259172	230.259172	250	0.92103669
5	6.43593951	657.43758	457.43758	257.43758	200	1.2871879
6	5.87518208	615.342073	448.675407	282.00874	166.666667	1.69205244
7	5.43936166	590.318539	447.461396	304.604253	142.857143	2.13222977
8	5.08805693	575.635644	450.635644	325.635644	125	2.60508515
9	4.79706608	567.61098	456.499869	345.388758	111.111111	3.10849882
10	4.55089647	564.071718	464.071718	364.071718	100	3.64071718
11	4.33910953	563.659821	472.75073	381.841639	90.9090909	4.20025803
12	4.15438109	565.487251	482.153918	398.820585	83.3333333	4.78584702
13	3.99140024	568.951779	492.028702	415.105625	76.9230769	5.39637313
14	3.84620952	573.632609	502.204037	430.775466	71.4285714	6.03085652
15	3.71579141	579.228302	512.561636	445.894969	66.6666667	6.68842454
16	3.59779956	585.518344	523.018344	460.518344	62.5	7.3682935
10.649	4.41004014	563.511202	469.605671	375.700139	93.905531	4.00083079
6.709	5.55607449	596.312651	447.25914	298.20563	149.05351	2.00066157

TABLE 3. SPREADSHEET ANALYSIS FOR FINDING THE HEIGHT OF AN OCTAGONAL PRISM WITH A FIXED VOLUME THAT GIVES MINIMUM TSA AND ESA.

This turns out to be true regardless of the number of sides. **Table 3** shows what you get if the bases are octagons.

Here the exposed surface area approaches a minimum when h is close to 6.71 and the ratio of the area of the base to the area of the walls is close to 2:1, and the total surface area approaches a minimum when h is close to 10.65, and the ratio of the area of the base to the area of the walls is close to 4:1.

Regular Anti-Prisms

Although anti-prisms are mentioned in most textbooks, few give formulas for their volume. In searching the Web for a formula I found the following slightly complicated but elegant way of calculating the volume of a regular anti-prism. It can be found at http://home.att.net/~numericana/answer/polyhedra.htm#antiprism and most of the calculations that follow are the work of Dr. Murali, V.R.D. The basic idea behind the method is that a regular n-gonal anti-prism can be constructed from a $2n$-sided regular prism by truncating $2n$ pyramids from the prism. For example, an 8-sided regular prism can be truncated to form a regular square anti-prism as shown in **Figure 2.** The volume of the anti-prism can be found by calculating the volume of the $2n$-gonal prism and subtracting from it $2n$ times the volume of one of the truncated pyramids.

Since we want to find the volume of an anti-prism given the length of a side, we need to find s, the length of an edge of the $2n$-gonal base of the prism and h, the height of the prism. For example, suppose we want to find the volume of a square anti-prism such as the one in **Figure 3.** You would start knowing the length of U_1U_2, an edge of the anti-prism and would need to calculate U_1U_2, the length of an edge of the prism and U_2L_2 the height of the prism. Note that the triangular faces of a regular anti-prism are equilateral.

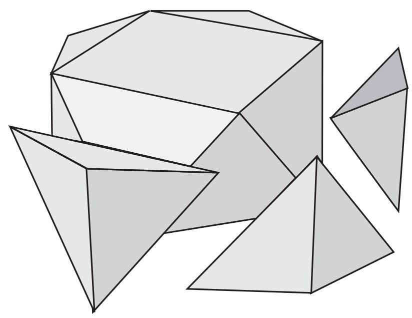

FIGURE 2. PARTIALLY TRUNCATED OCTAGONAL PRISM.

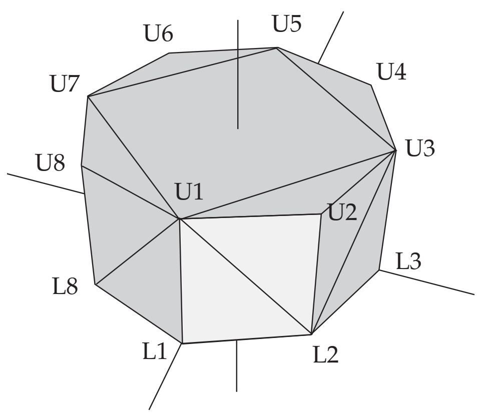

FIGURE 3. AN OCTAGONAL PRISM WITH OUTLINES OF PYRAMIDS TO BE TRUNCATED.

Figure 4 shows in more detail one of the octagonal bases of the octagonal prism. In what follows we will refer

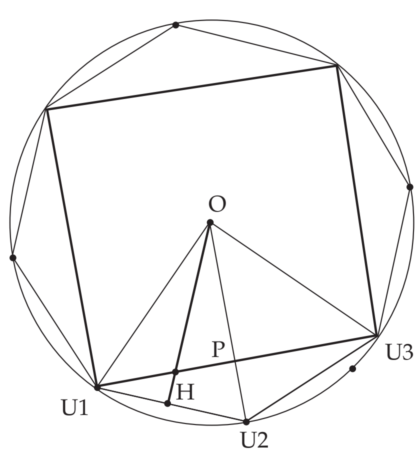

FIGURE 4.
THE BASE OF AN OCTAGONAL PRISM.

to this diagram in a general sense. For example, in the general case $\angle U_1OU_2 = \frac{360^\circ}{2n} = \frac{180^\circ}{n}$, while if $n = 4$ $\angle U_1OU_2$ would equal 45°.

1. Finding the edge length of the $2n$-gon base

The first thing you need to figure out is what the edge length would be for the $2n$-gon if the inscribed n-gon has edges of length a. In Figure 4 one of the bases of the $2n$-sided figure is shown along with the inscribed n-gon. If $U_1U_2 = a$, $\angle U_1OU_2 = \frac{360^\circ}{2n}$ and $\angle U_2U_1P = \angle HOU_2 = \frac{90^\circ}{n}$ then

$$U_1P = \frac{a}{2}$$

$$U_1U_2 = \frac{a}{2\cos\left(\frac{90}{n}\right)}$$

$$U_2P = \frac{a}{2}\tan\left(\frac{90}{n}\right)$$

$$OH = \frac{a}{4\sin\left(\frac{90}{n}\right)}.$$

Therefore, if the inscribed n-gon has edge length a, the circumscribed $2n$-gon has edge length equal to $\frac{a}{2\cos\left(\frac{90}{n}\right)}$.

Using the above, you can now show that

$$\begin{aligned}
\text{area}(2n\text{-gon}) &= (2n)\left(\tfrac{1}{2}\right)(U_1U_2)(OH) \\
&= n\left(\frac{a}{2\cos\left(\frac{90}{n}\right)n}\right)\left(\frac{a}{4\sin\left(\frac{90}{n}\right)n}\right) \\
&= \frac{na^2}{8\cos\left(\frac{90}{n}\right)\sin\left(\frac{90}{n}\right)} \\
&= \frac{na^2}{4\sin\left(\frac{180}{n}\right)}.
\end{aligned}$$

2. Finding the height of the $2n$-gonal prism

In order to get the volume of the $2n$-gonal prism, you now need to figure out its height. To do this, we'll use one of the truncated pyramids as shown in **Figure 5.**

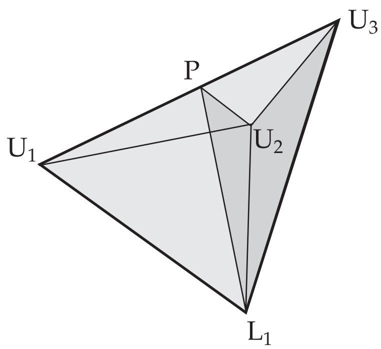

FIGURE 5. ONE OF THE TRUNCATED PYRAMIDS.

To find the height h note that since L_2PU_2 is a right triangle and $\Delta U_1L_2U_3$ is equilateral, you have

$$U_2L_2 = \frac{\sqrt{3}}{2}a,$$

and therefore

$$h = \sqrt{(U_2L_2)^2 - (U_2P)^2}$$
$$= \sqrt{\left(\frac{\sqrt{3}}{2}a\right)^2 - \left(\frac{a}{2}\tan\left(\frac{90}{n}\right)\right)^2}$$
$$= \left(\frac{a}{2}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}.$$

3. Finding the volume of the $2n$-gonal prism

volume($2n$-gonal prism) = area($2n$-gon) * h

$$= \left(\frac{na^2}{4\sin\left(\frac{180}{n}\right)}\right)\left(\frac{a}{2}\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}\right)$$
$$= \frac{na^3\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}}{8\sin\left(\frac{180}{n}\right)}$$

4. Finding the volume of the truncated pyramids

Next, you need to find the volume of one of the truncated pyramids. In Figure 5, $U_1U_2U_3L_1$ is a pyramid with height U_2L_1, and base $\Delta U_1U_2U_3$. In $\Delta U_1U_2U_3$, U_2P is the altitude to side U_1U_3 and therefore has area is equal to

$$\text{area}(\Delta U_1U_2U_3) = \left(\tfrac{1}{2}\right)(U_1U_3)(U_2P)$$
$$= \left(\frac{1}{2}\right)(a)\left(\frac{a}{2}\tan\left(\frac{90}{n}\right)\right)$$
$$= \frac{a^2\tan\left(\frac{90}{n}\right)}{4}.$$

This pyramid has height h so its volume is
volume(truncated pyramid) =

$$\left(\frac{1}{3}\right)\left(\frac{a^2\tan\left(\frac{90}{n}\right)}{4}\right)\left(\left(\frac{a}{2}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}\right)$$
$$= \left(\frac{a^3\tan\left(\frac{90}{n}\right)}{24}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}.$$

Since there are $2n$ of these pyramids, the volume of the anti-prism is

$$\text{volume} = \left(\frac{na^3}{8\sin\left(\frac{180}{n}\right)}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)} - \left(\frac{na^3}{12}\right)\tan\left(\frac{90}{n}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}$$
$$= \left(\frac{na^3}{24}\right)\left(\frac{3}{\sin\left(\frac{180}{n}\right)} - 2\tan\left(\frac{90}{n}\right)\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}.$$

This somewhat daunting formula simplifies nicely as follows. Working out the details is a good exercise and uses some fundamental trigonometric identities. If you let $t = \tan\left(\frac{90}{n}\right)$, then

$$\text{volume} = \frac{na^3\left(3 - t^2\right)^{\frac{3}{2}}}{48t}.$$

An even simpler version of this formula can be found by using

$$h = \left(\frac{a}{2}\right)\sqrt{3 - \tan^2\left(\frac{90}{n}\right)}.$$

In this case the volume of the anti-prism becomes

$$\text{volume} = \frac{nh^3}{6\tan\left(\frac{90}{n}\right)}.$$

After I finished writing this, I discovered another formula for the volume of a regular n-gonal prism at http://www.literka.addr.com/antiprisms.htm and will leave it as an exercise to show that this equivalent to

$$V = a^3\left(\frac{n}{12}\right)\left(\cot\left(\frac{90}{n}\right) + \cot\left(\frac{180}{n}\right)\right)\sqrt{1 - \left(\frac{1}{4}\right)\sec^2\left(\frac{90}{n}\right)}.$$ ❏

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.

1. Blackwell, William. *Geometry in Architecture*. 1984. John Wiley and Sons, Inc. p.129. ISBN # 0-471-09683-0

2. *ibid*, p.135

3. http://home.att.net/~numericana/answer/polyhedra.htm#antiprism