Last edition's Geometer's Corner contained several errors. For those interested, I have posted a corrected version of "Special Properties of Conics" on my Website, www.zebragraph.com.

CELL PHONE GEOMETRY

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I just purchased my first Smartphone and, as a result, have been fascinated by all that it can do. I became interested in the way it works, and, in the process, I discovered some very interesting applications of the geometry many of us teach.

What particularly interested me was how cell phones transmit and receive. I have often wondered about the antennas like the one in Figure 1 that have sprouted up all over in the past ten or so years. I wondered why they aren't taller, why there are so many of them, and why many have the triangular shape that has become so common. This column is an attempt to answer some of those questions. For those of you interested in how cellular technology evolved over the years, go to [3].

A cell phone is actually a miniature radio that both transmits and receives. It uses what is known as *duplex technology*, which means that you can talk and listen at the same time. This is not true for other communication devices such as walkie-talkies on which you have to press a button in order to talk. Cell phones don't have much power so they can only transmit a short distance. This is why there are so many cellular phone towers. Each tower defines what is known as a *cell*, hence the name, and each cell covers an area of roughly ten square miles. There are several different broadcast systems in use. Each tower has the ability to broadcast or receive on 832 different frequencies, as does your phone. In order to avoid interference problems, no two adjacent cells can use the same set of frequencies. One broadcast system assumes that the region served by a tower is hexagonal in shape. Ideally the broadcast regions would form a hexagonal grid like the one in **Figure 2**. In order to avoid interference, the hexagons numbered 1 through 7 must use a separate set of frequencies.

As shown in Figure 2, each cell has six neighboring cells, so to avoid interference there must be seven distinct sets of frequencies in use. Cells 1 through 7 must use different sets of frequencies but cell 8 can use the same set as cell 1, as can cell 9. Since there are 832 frequencies available for cell phone use, 42 of which are used for control purposes, there are 790 frequencies available for users, but duplex voice channels require two frequencies each, leaving 395 available duplex voice channels. This means that each of the seven cells, six adjacent to cell 1, can only handle up to 56 users talking at once [1]. In short, there tend to be lots of small cells, which means lots of broadcast towers. Because of the small size of the cells, the transmission and receiving devices in a cell phone don't require lots of power—they do not transmit over great distances; this means the batteries in a phone can be smaller and will last longer.

As you travel and move from cell to cell, a lot goes on of which you are unaware. Without your knowing it, your phone will change frequencies from the pair it is using in the cell you are currently in to the pair it will use in the cell you are entering. In order to do this, all the towers in a large area are connected to what is known as the Mobile Telecommunications Switching Office. The MTSC keeps track of each call being processed, aware of when there needs to be a frequency change, or as it is known in the trade, a *hand off*. One of the reasons it has taken so long for cellular technology to become available is that the hand-off process requires huge amounts of computing power that in turn require huge storage capabilities that only recently have become available. To make the system work, the cells have to be carefully defined and controlled, which is the subject of the rest of this column.

FIGURE 1. A TYPICAL CELLULAR TELEPHONE NETWORK ANTENNA.

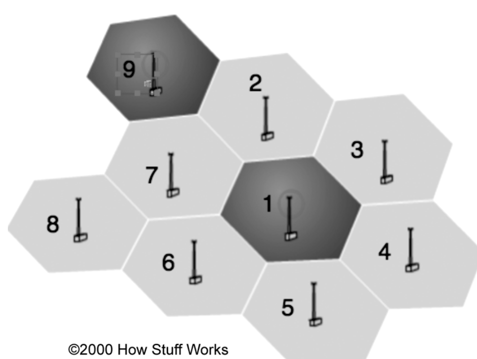

FIGURE 2. HEXAGONAL ARRANGEMENT OF CELLULAR BROADCAST REGIONS.

In order to provide uninterrupted service as you move, your phone is constantly checking the strength of the channel provided by the cell you are in and comparing it to the strength of a neighboring cell. As soon as the neighboring signal is stronger, your phone is automatically switched over to that channel. This also means that you have moved to a different cell. In setting up a cell service it is very important that companies have an idea of what the shapes of the cells are, given cell tower locations. So, this is the geometric problem: Given a collection of tower locations, what are the shapes of the cells or broadcast regions they define?

If Earth was perfectly flat and there were no vertical obstructions like buildings, the answer is easy. The cells would all be regular hexagons as shown in Figure 2. Earth is not flat, and for a variety of reasons the towers cannot be placed in such a way as to produce regular hexagonal cells.

One way of answering what the cell shapes are is to determine, given a set of tower locations, what are known as the Voronoi, or nearest neighbor regions. A nearest neighbor region for a tower contains all points that are closest to that tower.

In order to answer the cell-shape question, let's start with the simplest case. Suppose there are only two towers in the network, say T_1 and T_2. The perpendicular bisector of the segment T_1T_2 is the set of all points that are equidistant from T_1 and T_2, so all the points on the same side as T_1 of the bisector as T_1 are closer to T_1, and those on the other side are closer to T_2.

If there are three non-collinear towers T_1, T_2 and T_3, then, using the same approach, you need to draw the perpendicular bisectors of the sides of triangle $T_1T_2T_3$. These intersect at the circumcenter C of the triangle. **Figure 3** shows the nearest neighbor regions for a three-tower set up.

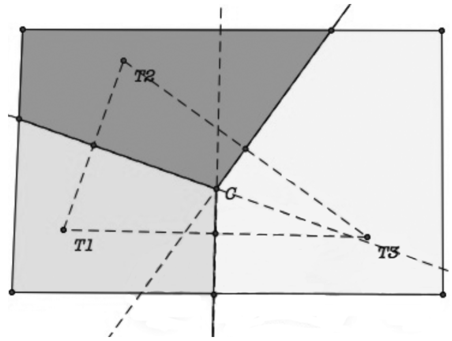

FIGURE 3. NEAREST NEIGHBOR REGIONS FOR THREE TOWERS.

When there are more than three towers, the nearest neighbor regions are much harder to determine. For a long time there was no known easy-to-use algorithm for finding Voronoi Regions. However, in 1986, Steven Fortune of Bell Labs came up with a new algorithm that is much faster and much easier to implement. The algorithm makes use of the following geometric definition of a parabola.

Given a line in the plane and a point not on the line, the set of points that are equidistant from the point and the line forms a parabola. The line is called the directrix and the fixed point is called the focus.

Suppose the focus has coordinates (x_f, y_f), and the directrix is the horizontal line $y = y_d$; then the definition above gives us

$$\sqrt{(x-x_f)^2+(y-y_f)^2} = |y-y_d|$$
$$(x-x_f)^2+(y-y_f)^2 = |y-y_d|^2$$
$$x^2-2xx_f+x_f^2+y^2-2yy_f+y_f^2 = y^2-2yy_d+y_d^2$$
$$2yy_f-2yy_d = x^2-2xx_f+x_f^2+y_f^2-y_d^2$$
$$2y(y_f-y_d) = x^2-2xx_f+x_f^2+y_f^2-y_d^2$$
$$y=\left(\frac{1}{2(y_f-y_d)}\right)\left(x^2-2xx_f+x_f^2+y_f^2-y_d^2\right)$$

Formula 1.

This is the function we will use in implementing Fortune's Algorithm with a geometric construction package such as the *Geometer's SketchPad*. The construction I will describe is based on one I found in David Austin's American Mathematical Society Feature Column [2]. This site contains an applet that will create Voronoi Regions using the following process.

To implement Fortune's Algorithm on a rectangular region and a set of points such as those in **Figure 4**, we will need to generate a set of parabolas which have different focal points but the same directrix. Fortune calls the movable directrix the *sweep line*. He noticed that something very interesting happens when you start with the sweep line above all the points in a defined region, and then move it until it is below the lowest point, keeping it parallel to the x-axis. Note that any point *above the sweep line* defines a parabola that opens up, and any point *below the sweep line* defines one that opens down. Fortune noticed that the parabolas that open up intersect in interesting ways. To be specific:

- If two parabolas intersect in a point, that point has to be equidistant from the foci of the two intersecting parabolas. As the sweep line moves down, the intersection of the two parabolas traces out a set of points that are equidistant from the two focal points, and thus lay on a boundary of a two nearestneighbor regions.
- If three parabolas intersect in a point, that point has to be equidistant from the three focal points and is the intersection of the boundaries of three regions.
- Finally, and probably most importantly, Fortune noticed that the intersections that matter occur along the lower boundary formed by the parabolic arcs. He calls this the *beach line*.

What follows is how I used the *Geometer's SketchPad* to implement Fortune's Algorithm. I also used a drawing program to add detail to the GSP sketches. Consider the rectangular region shown in Figure 4 with five points.

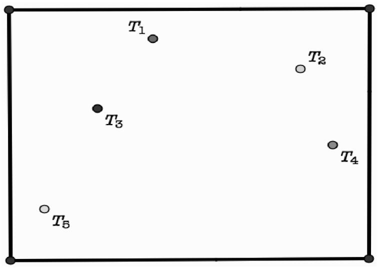

FIGURE 4. FIVE POSSIBLE TOWER LOCATIONS.

In terms of the cell phone problem, think of these points as towers. To get the nearest neighbor regions for the five tower locations in Figure 4, do the following:

Step 1. Construct a movable horizontal line just below T_1. This is the *sweep line*.

Step 2. Using Formula 1, construct the parabola with focus T_1 and directrix the *sweep line* (**Figure 5**).

Step 3. Create the parabola with focus T_2 and directrix the *sweep line*.

This creates a second parabola as shown in **Figure 6**.

Notice that if the sweep line lies below T_1 but above T_2, you get two parabolas that open in opposite directions, though we are only interested in parabolas that open in the same direction. If you move the *sweep line* below T_2 as shown in **Figure 7**, things get interesting. Notice that now the two parabolas intersect at two points P_1 and P_2 (P_2 is not shown because it is off the screen). The two parabolas will always intersect in two points, but since we are only considering nearest neighbor regions, we will ignore the point off the screen as being too far away. Let F be the foot of the perpendicular from P_1 to the sweep line. Since $PT_1 = PF$ and $PT_2 = PF$, P_1 is equidistant from T_1 and T_2. In a similar fashion you can show that P_2 is equidistant from T_1 and T_2. Fortune refers to the points where the parabolic arcs forming the beach line intersected as *break points*.

Since P is equidistant from T_1 and T_2, the nearest-neighbor boundary line for T_1 and T_2 is simply the perpendicular bisector of T_1T_2.

Next we will introduce a third parabola with FOCUS T_3, as shown in **Figure 8**. Note that as we move the sweep line down, the parabolas change shape, as does the beach line. Eventually, the sweep line will be below T_3, and the parabola defined by T_3 will open upward. When this happens, parabola T_3 will intersect parabola T_1 in two points, each of which is equidistant from T_1 and T_3. The trace of these points is the perpendicular bisector of segment T_1T_3. As the sweep line moves farther down, parabola T_3 will intersect parabola T_2 in two points that lie on the perpendicular bisector of T_2T_3.

As the sweep line moves down and below T_4, another parabola is formed, and new breakpoints appear on the beach line (**Figure 9**). Tracing these break points creates more of the nearest neighbor boundaries.

As the sweep line moves farther down, parabolas T_1, T_2, and T_3, eventually meet at a single point. Since the points T_1, T_2 and T_3 are the vertices of a triangle, the three perpendicular bisectors meet at a point, the circumcenter of the triangle $T_1T_2T_3$. Fortune calls this a *circle event*, and the point determines a shared vertex for three nearest neighbor regions. **Figures 10–12** show what happens as the sweep line is moved down, and **Figure 13** shows the resulting Voronoi regions.

The preceding represents one way of determining the regions covered by each tower. This turns out to provide a decent approximation; there is a

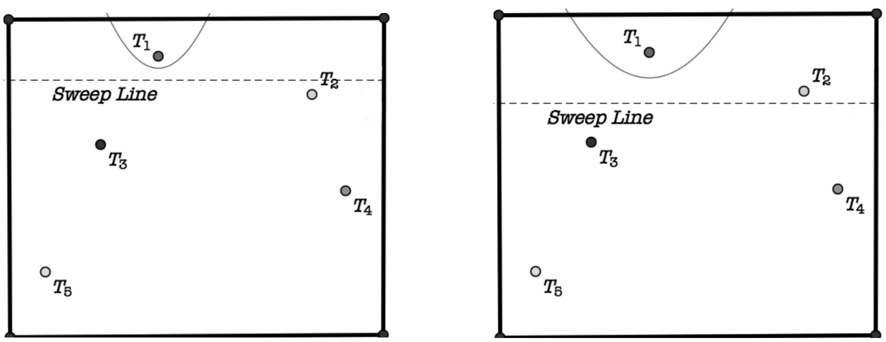

FIGURE 5. PARABOLA T_1 CHANGES SHAPE AS THE SWEEP LINE MOVES DOWN.

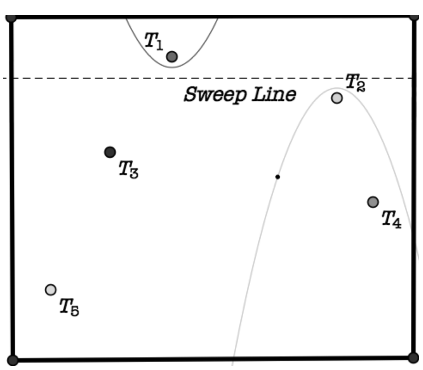

FIGURE 6. PARABOLAS T_1 AND T_2.

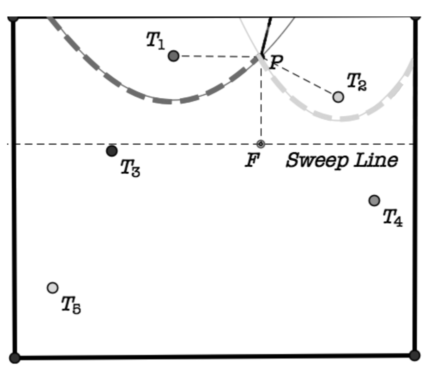

FIGURE 7. PARABOLAS T_1 AND T_2 INTERSECTING AT A BREAK POINT P ALONG THE BEACH LINE.

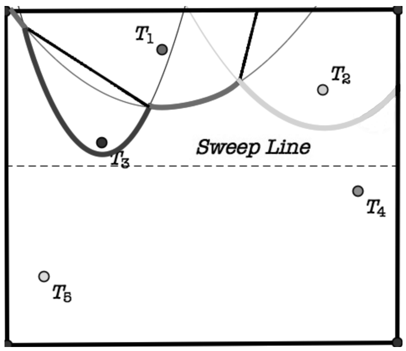

FIGURE 8. THE *SWEEP LINE* HAS MOVED BELOW T_3, AND PARABOLA T_3 HAS BEEN CREATED. THE BEACH LINE CONSISTS OF THREE PARABOLIC ARCS, AND THE BOUNDARY BETWEEN T_1 AND T_3 HAS APPEARED.

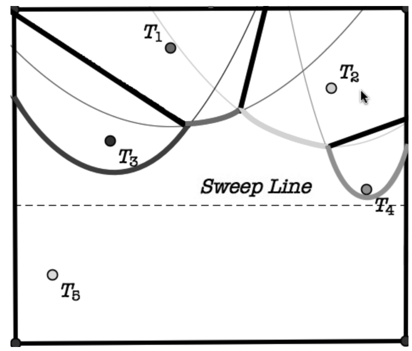

FIGURE 9. THE SWEEP LINE HAS MOVED BELOW T_4. THE BEACH LINE NOW CONSISTS OF FOUR PARABOLIC ARCS, AND THE BOUNDARY BETWEEN T_2 AND T_4 HAS APPEARED.

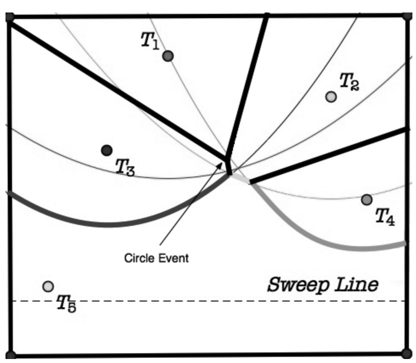

FIGURE 10. THE SWEEP LINE IS JUST PAST T_5. THE BEACH LINE CONSISTS OF THREE PARABOLIC ARCS, AND THE BOUNDARY BETWEEN T_3 AND T_5 HAS APPEARED.

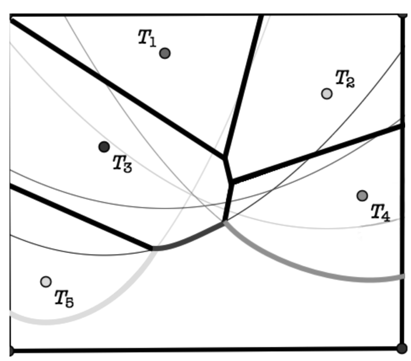

FIGURE 11. THE SWEEP LINE HAS MOVED FARTHER DOWN AND IS NOW OUT OF THE FRAME. THE BEACH LINE CONSISTS OF THREE PARABOLIC ARCS, AND THE BOUNDARY BETWEEN T_3 AND T_4 HAS APPEARED.

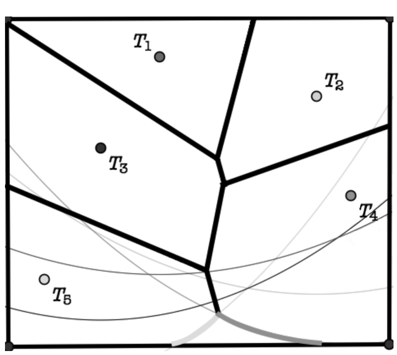

FIGURE 12. THE SWEEP LINE HAS MOVED EVEN FARTHER OUT OF THE FRAME. THE BEACH LINE CONSISTS OF TWO PARABOLIC ARCS, AND THE BOUNDARY BETWEEN T_4 AND T_5 HAS APPEARED.

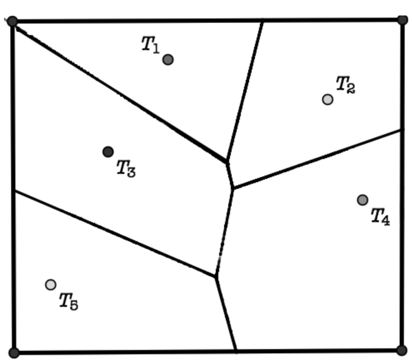

FIGURE 13. THE FINAL DIAGRAM WITH ALL THE PARABOLAS REMOVED.

method in use that is similar to this one that uses what are known as *weighted Voronoi Regions*. It should also be noted that the nearest neighbor regions should use a metric that takes into account signal strength that is measured using an inverse square law. For more on this, go to [4].

My original plan for this column was to find a map showing where the towers are in the area where I live and use that as the source for the diagram explaining Fortune's Algorithm. Unfortunately, this is much easier said than done. Many towers have multiple antennas, and it is not easy to determine which antenna belongs to which company. Also, any tower less than 200 feet in height does not have to be federally registered; so there is no actual record of what towers and antennae belong to which company. It can be done, but it would take a while. However, once I get an accurate map of the towers in my area for a given provider, I will use it as a basis for a project using Fortune's Algorithm. ❏

References

[1] Brain, Marshall, Jeff Tyson and Julia Layton. How Cell Phones Work. http://electronics.howstuffworks.com/cell-phone.htm.

[2] Austin, David. Voronoi Diagrams and a Day at the Beach. http://www.ams.org/featurecolumn/archive/voronoi.html.

[3] Farley, Tom. The Cell-Phone Revolution. http://www.americanheritage.com/events/articles/web/20070110-cell-phone-att-mobile-phone-motorola-federal-communications-commission-cdma-tdma-gsm.shtml.

[4] Hill, David R. Cell Phones: Signal-to-Interference Ratio. http://mathdemos.gcsu.edu/mathdemos/cellsir/cellsir.html#Objectives.