This issue's Geometer's Corner is written by two students, Cynthia Liu and Soonkyu Park, who are taking a tutorial with me this year. We have been taking an intensive look at the conics and making extensive use of Cabri 3D and GeoGebra in our work. For the past month, we have been studying a new book that the American Mathematical Society is publishing: *Geometry of Conics* by A.V. Akopyan and A. A. Zaslavsky. What Cynthia and Soonkyu have written will give you a feel for the book. It is an ideal book for a senior level high school elective for students who have studied the conics and would like to study more geometry. It is also a great book for teachers to learn more about the conics and, in the process, learn some useful mathematical techniques, such as inversion. All the figures in the article were made with GeoGebra, a free computer based geometric construction package that has many useful features lacking in Cabri 2 and the Geometer's SketchPad. I have the actual GeoGebra files that were used on my Website, www.zebragraph.com. Please feel free to download them.

— Jonathan Choate

SPECIAL PROPERTIES OF CONICS

CYNTHIA LIU AND SOONKYU PARK

Introduction— A Point Inside and Out

For all three conics, there are special properties associated with a point located inside (or outside of) the curve and the distances from the point to the focus (or foci) and/or to the directrix(es) of the conic.

In the case of an ellipse, the sum of the distances from any point inside the ellipse to the foci is less than the length of the major axis. If the point is located outside, the distance is longer than the major axis.

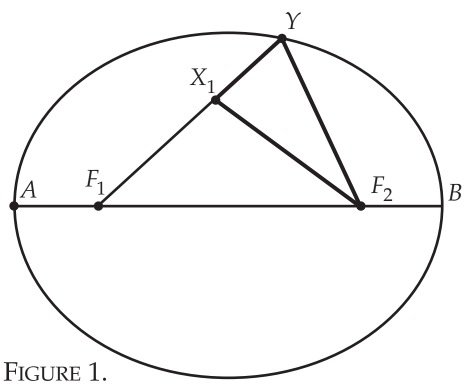

FIGURE 1.

To prove this, connect a point X_1 inside the ellipse to one of the foci, and extend this line until it intersects the ellipse at a point Y (**Figure 1**). Applying the triangle inequality to triangle X_1F_2Y, we get that $X_1F_2 < X_1Y + YF_2$. Adding F_1X_1 to both sides gives us $F_1X_1 + X_1F_2 < F_1X_1 + X_1Y + YF_2 = F_1Y + F_2Y$. Likewise, for a point X_1 outside the ellipse, $F_2Y < X_1Y + X_1F_2$, and, therefore, $F_1X_1 + X_1F_2 = F_1Y + YX_1 + X_1F_2 > F_1Y + F_2Y$. Now, to show that $F_1Y + F_2Y$ is equivalent to AB, the major axis, we place the point Y on the intersection between the major axis and the ellipse. Due to the symmetry within the ellipse, we know that F_1A and F_1B are equal. Therefore, $F_1Y + F_2Y = F_1A + F_2A = F_1A + F_1F_2 + F_2B$.

The opposite holds true for a point outside the ellipse.

In the case of a hyperbola, the absolute value of the difference of the distances from any point between the two branches to the foci is greater than that of the distances from any point on the hyperbola to the foci.

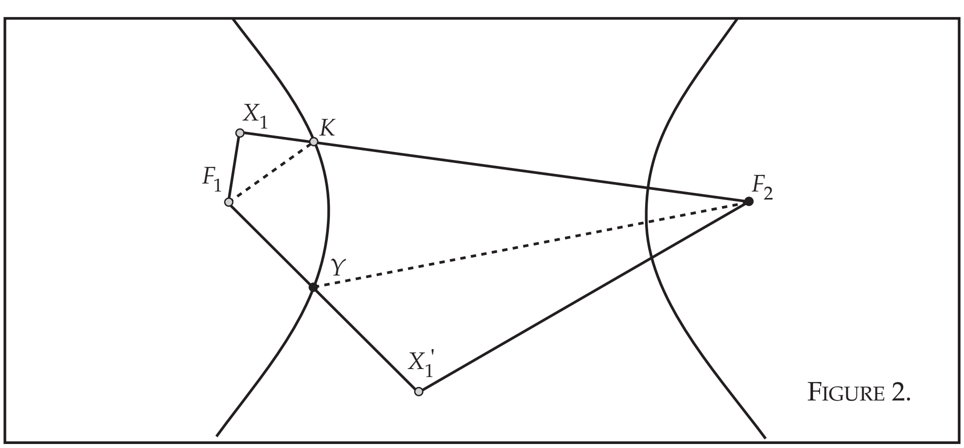

FIGURE 2.

To prove this, let X_1' be a point between the two branches of the hyperbola, and Y the intersection of $X_1'F_1$ and the hyperbola (**Figure 2**). Then, $F_1X_1' = F_1Y + YX_1'$. By the triangle inequality, $F_2X_1' < F_2Y + YX_1'$. Subtracting F_1X' from both sides, we have $F_2X' - F_1X' < F_2Y + Y'X' - (F_1Y + YX') = F_2Y - F_1Y$.

In a similar fashion, we can show the opposite is true for a point inside one of the branches of the hyperbola.

In the case of a parabola, the distance between a point inside the parabola and the focus is less than the distance between that point and the directrix.

To prove this, let X_0 be the point, F the focus, and Z the projection of X_0 onto the directrix (**Figure 3**). $X_0F < X_0Z + FZ$ by the triangle inequality. Since $FZ = ZY$ by the definition of the parabola, $X_0F < X_0Z + ZY = X_0Y$.

For points outside the parabola, the opposite holds. Let X' be such point, F the focus, and Z the projection of X' onto the directrix. By the triangle inequality, $X'F + XZ' > FZ' = Z'Y' = Z'X' + X'Y'$. Subtracting $X'Z'$ from both sides yields $X'F > X'Y'$.

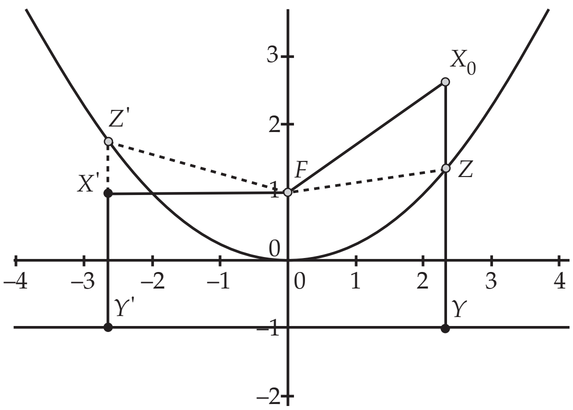

FIGURE 3.

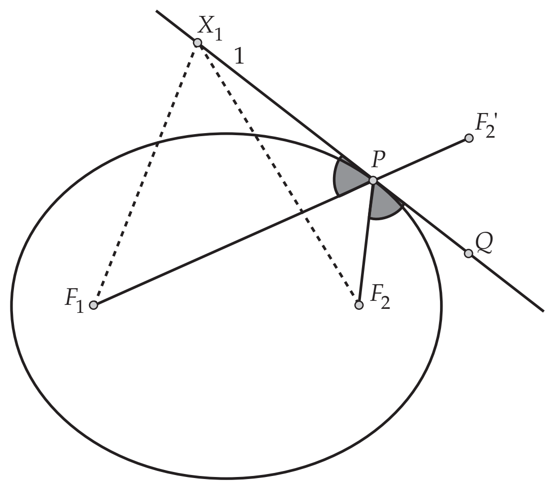

FIGURE 4.

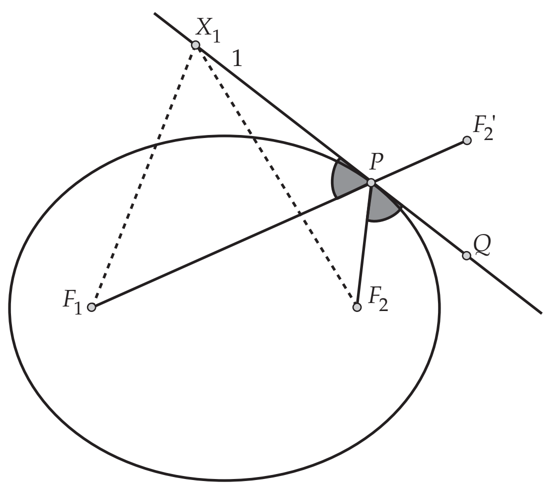

FIGURE 5.

Optical Properties

Optical properties describe the relationship between a line tangent to a conic and the exterior angle formed by the tangent.

In the case of an ellipse, a line l that is tangent to the ellipse at point P is the bisector of the external angle $\angle F_1PF_2$ (**Figure 4**).

To prove this, draw a line tangent to the ellipse and pick a point X_1 that lies on the line and is outside the ellipse. Straight away, we can tell that $X_1F_1 + X_1F_2 > PF_1 + PF_2$. Because the point of tangency provides the smallest sum of F_1P and F_2P, we can deduce that $\angle X_1PF_1$ and $\angle QPF_2$ must be congruent. This makes sense because if we reflect F_2 over the line, F_1F_2' has the smallest length when the three points (F_1, P, and F_2') are collinear. In order for that to happen, $\angle X_1PF_1$ and $\angle QPF_2'$ must be congruent since they are vertical angles. Therefore, because $\angle QPF_2'$ and $\angle QPF_2$ are congruent (property of a reflected point), $\angle X_1PF_1$ and $\angle QPF_2$ are congruent.

In the case of a hyperbola, a line that is tangent to the hyperbola at point P is the bisector of $\angle F_1PF_2$, where F_1 and F_2 are the foci of the hyperbola (**Figure 5**).

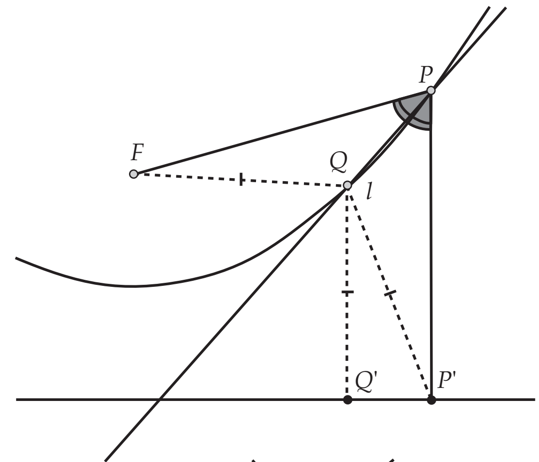

FIGURE 6.

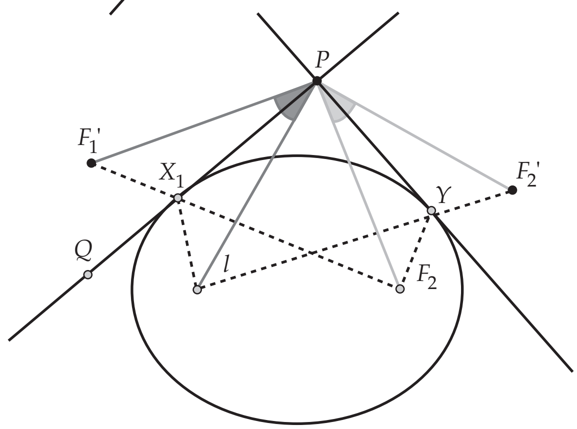

FIGURE 7.

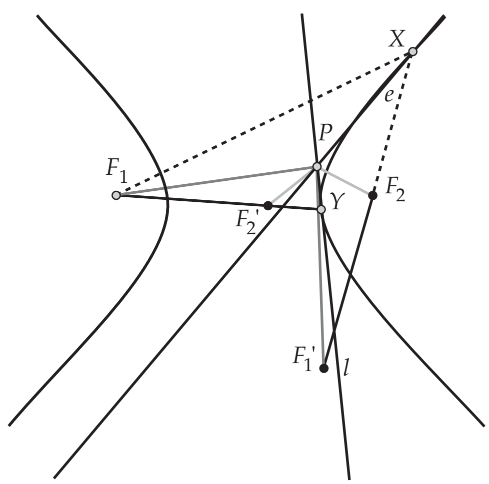

FIGURE 8.

To prove this, we use proof by contradiction. Suppose the bisector of $\angle F_1PF_2$ intersects the hyperbola twice at P and Q. Reflect F_1' with respect to l and call the second point F_1'. Because l is the angle bisector of $\angle F_1PF_2$, $F_1Q = QF_1'$ and $F_1P = PF_1'$. Hence, F_2, F_1', and P are collinear. By definition of a hyperbola (that is, the difference between the distances from the foci to any point on the hyperbola is constant), $F_2P - PF_1 = F_2Q - F_1Q$, and, thus, $F_2F_1' = F_2P - PF_1 = F_2Q - QF_1'$. But, by the triangle inequality, $F_2F_1' > F_2Q - QF_1'$. This is a contradiction, because line l should be an angle bisector. Therefore, the line intersects the hyperbola at only one point and must be a tangent.

In the case of a parabola, a line l that is tangent to a parabola at point P is the angle bisector of $\angle FPP'$, where P' is the projection of P on the directrix (**Figure 6**).

To prove this, we once again use proof by contradiction. Suppose the bisector of $\angle FPP'$ intersects the parabola at two points, P and Q. Let Q' be the projection of Q onto the directrix. By the definition of the parabola, $PP' = PF$ and $QQ' = QF$. Since edge PQ is shared, $PP' = PF$, and $\angle FPQ = \angle P'PQ$, triangle $PP'Q$ is congruent to triangle PFQ (SAS congruence). Thus $QP' = QF = QQ'$. But this is impossible because, by the definition of a parabola, Q' is the only point where its distance from Q is equal to the distance from Q to F. Therefore, the bisector has to go through only one point on the parabola; that is, it is tangent to the parabola at P.

Isogonal Properties

The optical properties of conics can be used to produce proofs of the following results.

In the case of an ellipse, $\angle F_1PX_1$ and $\angle F_2PY$ are equal, where P is any point outside the ellipse and X and Y are the points of tangency from P to the ellipse (**Figure 7**). This is called the isogonal property of an ellipse.

To prove this, we start by noting that $\angle F_1'PX_1$ is congruent to $\angle F_1PX_1$, and that $\angle F_2'PY$ is congruent to $\angle F_2'PY$ (property of a reflected point), so $\angle F_1PX_1$ and $\angle F_2PY$ are congruent if $\angle F_1'PF_2$ and $\angle F_1PF_2'$ are congruent. In other words, the above theorem is established if we can prove triangles PF_2F_1' and PF_1F_2' congruent. It is easy to see that segments $F_1'P$ and F_1P are congruent (property of a reflected point) and that PF_2 and PF_2' are congruent, but the third congruency is a little more complicated. According to the Optical Property, $\angle F_2X_1P$ is equivalent to $\angle F_1X_1Q$, and because $\angle F_1X_1Q$ and $\angle F_1'X_1Q$ are congruent