

# NEW TECHNOLOGY MEETS SOME OLD GEOMETRY

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I have just spent the better part of two weeks learning to use a magnificent new piece of software that I saw demonstrated last month at the Anja Greer Conference on Math, Science and Technology, held annually at Phillips Exeter Academy [1], a wonderful conference that is about to celebrate its twenty-fifth year. The software, *Geometry Expressions (GE)* [2], is a geometric construction package that has some unique and useful features. While learning to use the software, I came across several sources of interesting exercises: two books, *How Round is Your Circle: Where Engineering and Mathematics Meet* [3] and *Mathematics Meets Technology* [4], and two Websites, Go Geometry Step-by-Step From the Land of the Incas [5], as complete a collection of interesting Euclidian geometry programs as any I have found on the Internet, and Cut-The-Knot [6], a wonderful collection of mathematical content appropriate for all levels of the mathematics curriculum, almost all of which contain easy-to-use instructional applets.

For years, since long before any technology other than pencil or paper existed, I have used a problem similar to the one below when I teach geometry.

**Problem:** Given tangent circles M and N with radii 8 and 4, respectively, find the length of a common tangent.

**Figure 1** is a screenshot of the figure that I created to solve the problem. I have added brief descriptions of the four toolbars on the right side of the screen and their purposes.

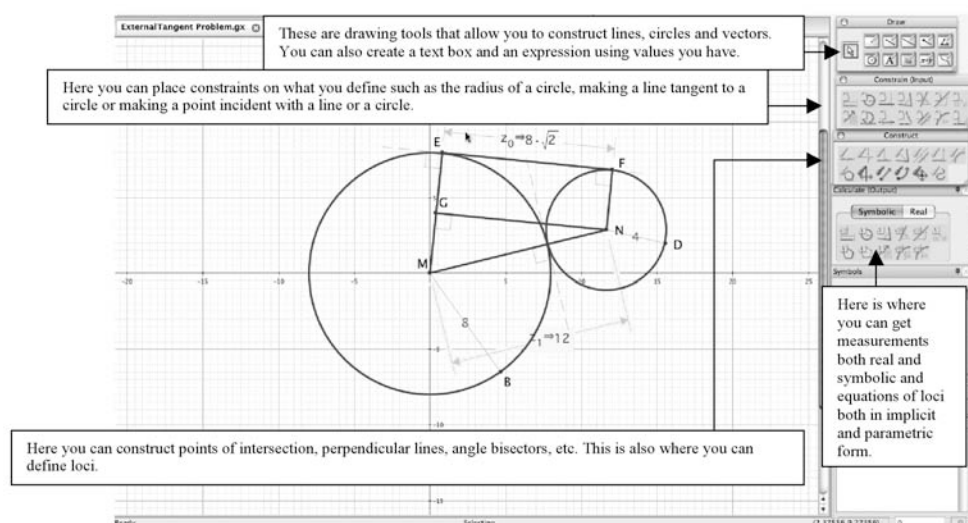


FIGURE 1. A GEOMETRY EXPRESSIONS SOLUTION.

Here is how I used *GE* to create the diagram in Figure 1.

1. Using the DRAW circle tool, construct a circle and then use the CONSTRAIN tool to make its radius 8.
2. Use the DRAW circle tool to construct a circle, the constrain RADIUS tool to make its radius 2, and the tangency tool to make it tangent to the circle constructed in step 1.
3. Use the DRAW segment tool to create a segment EF with E on circle M and F on circle N. Use the tangency tool to make the segment tangent to circle M at E and then tangent to circle N at F.
4. Construct segment NG using the incidence tool to place G on ME, and then the CONSTRUCT perpendicular tool to make NG perpendicular to ME.
5. Finally, use the CALCULATE Symbolic tool to measure EF and MN.

You get an interesting result if you do the problem slightly differently taking advantage of *GE*'s ability to make symbolic computations. *GE* allows you to use variables to define lengths and angles. Suppose you define the radius of circle M to be  $r$  instead of 8 and the radius of circle N to be  $s$  instead of 4 and then use steps 3–5 as given above. Your figure will now look like **Figure 2**.

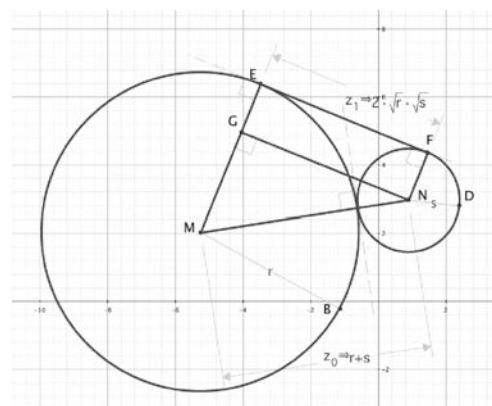


FIGURE 2. THE PROBLEM SOLVED SYMBOLICALLY.

This version of the solution gives a neat proof of an important result. Since EFGN is a rectangle,  $GN = EF$ . Triangle MGN is a right triangle with hypotenuse MN. *GE* tells us that  $GN = 2\sqrt{r}\sqrt{s}$  and  $MN = r + s$ . Proving this makes use of the Pythagorean Theorem and is a good exercise for any geometry class. Since  $GN < MN$  we have that

$$2\sqrt{r}\sqrt{s} = \sqrt{rs} < r + s$$

$$\sqrt{rs} < \frac{r+s}{2}$$

But  $\sqrt{rs}$  is the geometric mean of  $r$  and  $s$  and  $\frac{r+s}{2}$  is the arithmetic mean, so we have just come up with a nice proof that the arithmetic mean of two numbers is always greater than their geometric mean.

*GE* can be used to create and play with many interesting loci. What makes this software unique is that if you define a locus symbolically, *GE* can produce the equation of the locus in either parametric or implicit form. Here are two examples.

## Constructing an Ellipse

The following construction is from Alex Bourgogne's Cut-The-Knot Website [7]. In **Figure 3**, circles A and D have fixed radii and positions: Circle A is centered at the origin with radius 8, circle D is centered at  $(-8, 0)$  with radius 2. Circle F is variable and tangent to both circle A and circle D.  $\angle\theta = \angle DAF$ . As  $\theta$  goes from 0 to  $2\pi$ , circle F moves and its center F generates a curve that looks surprisingly like an ellipse. *GE* can produce the implicit form of the curve's equation, which can then be shown to be the equation of an ellipse.

*GE* identifies the implicit equation as  $Z_0$ . Doing a little algebra we get

$$-40x + 5x^2 + 9y^2 = 100$$

$$5(x^2 - 8x + 16) + 9y^2 = 180$$

$$\frac{5(x-4)^2}{180} + \frac{9y^2}{180} = 1$$

$$\frac{(x-4)^2}{36} + \frac{y^2}{20} = 1$$

which is the equation of an ellipse with a major axis of length 12 and a minor

axis of length  $2\sqrt{20}$ . By the way, you can also use *GE*'s COPY AS feature to copy equation  $Z_0$  and paste it into any of the popular computer algebra systems such as Mathematica, Maple, or Texas Instruments's new Nspire system then analyze the equation there. For those of you who want to pursue this problem further, you might note that  $36 = 6^2$ , 6 is the arithmetic mean of 2 and 10 (the radii of the circles used in the construction), and 20 is the square root of the geometric mean of 2 and 10.

## Analyzing Linkages 1

A mechanical linkage is "a system of links; especially a system of links or bars which are jointed together and more or less constrained by having a link or links fixed, and by means of which straight or nearly straight lines or other point paths may be traced." [8]

A type of linkage that you see often is the one used on the windshield wipers of buses, which is a special case of what are known as 4-bar linkages. In **Figure 4**, A and B are fixed points, AE and BF are two rods of equal length, and EF is a rod with the same length as AB. Rod AE is connected to a motor that rotates it about A. As AE rotates, point E moves along a circle, as does point F. A wiper *JI* is rigidly attached to EF at G, its midpoint. Since ABEF is a parallelogram, EF is always parallel to AB. Since *JI* is always perpendicular to EF, it will always be perpendicular to AB. As AE rotates about A, the wiper blade sweeps out the area bounded by PQ, arc PO, ON, and arc NQ. In building this model, *GE* allowed me to define the lengths of all the rods as variable quantities that I could change in the variable window shown to the right of the windshield model. The locus generated by points J and I on the wiper blade were created by using *GE*'s Locus function and allowing the value of  $\theta$ , which equals angle BAE to vary between 0.8 radians and 2.4 radians. The Variables window directly below the Edit Locus window

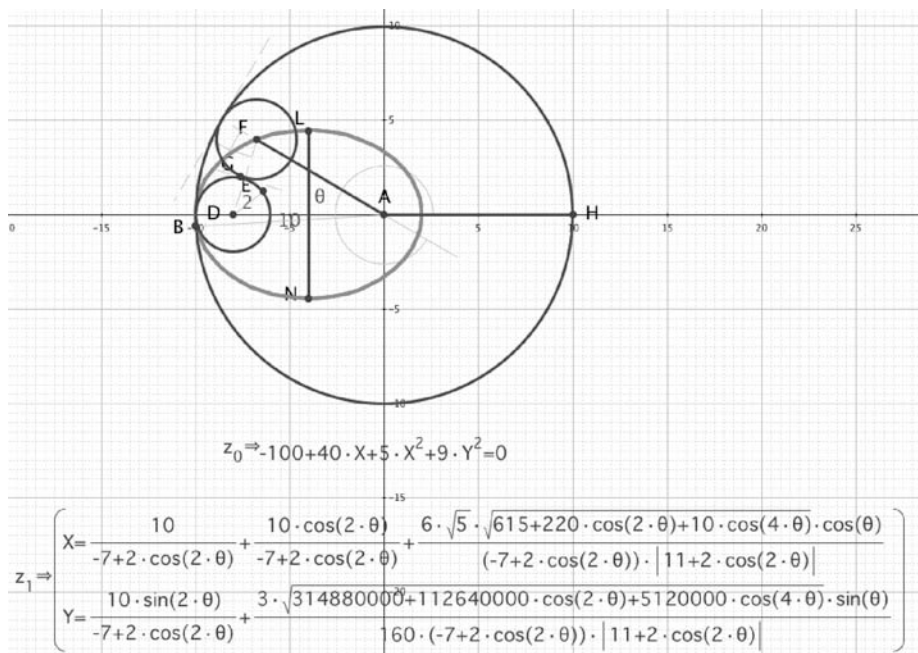


FIGURE 3. USING TANGENT CIRCLES TO CONSTRUCT AN ELLIPSE.

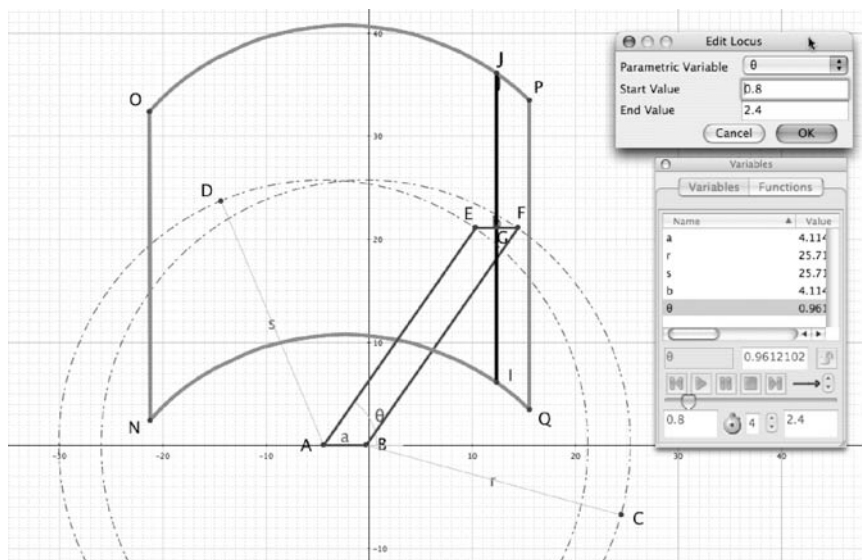


FIGURE 4. A SPECIAL CASE OF A 4 BAR LINKAGE.

shows the current values of  $a$  (the length of  $AB$ ),  $b$  (the length of  $EF$ ),  $r$  (the length of  $BF$ ), and  $s$  (the length of  $AE$ ). The beauty of what I created is that I can change any of these values and see the new locus, giving me a tool for studying in depth the behavior of 4-bar linkages with one bar rigidly located. For more on linkages and how they explain the workings of car jacks, children's swings, window openers, bi-fold doors—just to mention a few—see Brian Bolt's *Mathematics Meets Technology* [4].

## Analyzing Linkages 2

This excerpt from Wikipedia speaks to the historical significance of linkages:

*The Industrial Revolution was the golden age of mechanical linkages. Mathematical, engineering and manufacturing advances provided both the need and the ability to create new mechanisms. Many simple mechanisms that seem obvious today required some of the greatest minds of the era to create. Leonhard Euler was one of the first mathematicians to study linkage synthesis, and James Watt worked very hard to invent the Watt linkage to support his steam engine's piston. Chebyshev worked on mechanical linkage design for*

*over thirty years, which led to his work on polynomials. New linkage inventions, designed by need, were instrumental in cloth making, power conversion and speed regulation. Even the ability of a mechanism to produce accurate linear motion, without a reference guide way, took years to solve.* [9]

The book *How Round is Your Circle* gives many examples of linkages that were used during the late nineteenth

century to mechanically produce both approximations for a variety of different curves and exact representations. **Figure 5** shows a model of a linkage developed by the Russian mathematician Pafnuty Chebyshev (1821–94) that converts circular motion into linear motion [10].

$A(-2, 0)$  and  $B(2, 0)$  are fixed points.  $AC$  is a rod of length 5 that rotates around  $A$ .  $BE$  is a rod of length 5 that rotates around  $B$ .  $EC$  is a rod of length 2.  $F$  is the midpoint of  $EC$ . As  $AC$  rotates, point  $E$  starts to move along circle  $B$  and the rod  $EC$  moves. The model in Figure 4 was built using  $GE$ , and it appears that the locus of point  $F$  as  $AC$  rotates is a straight line. When I built the model I defined  $\angle ABC$  as  $\theta$  so that the coordinates of point  $F$  could be found in terms of  $\theta$ . I transferred the parametric equations to Mathematica and did a plot of them. **Figure 6** shows the equations in parametric form and their plot. Notice that what appears to be a straight line is in fact slightly curved, because as  $x$  goes from  $-2$  to  $2$ ,  $t$  oscillates between 4 and 4.01, a deviation of .01 units. This ability to define a locus in terms of a parameter and then have the software generate the equations that can then be analyzed in a computer

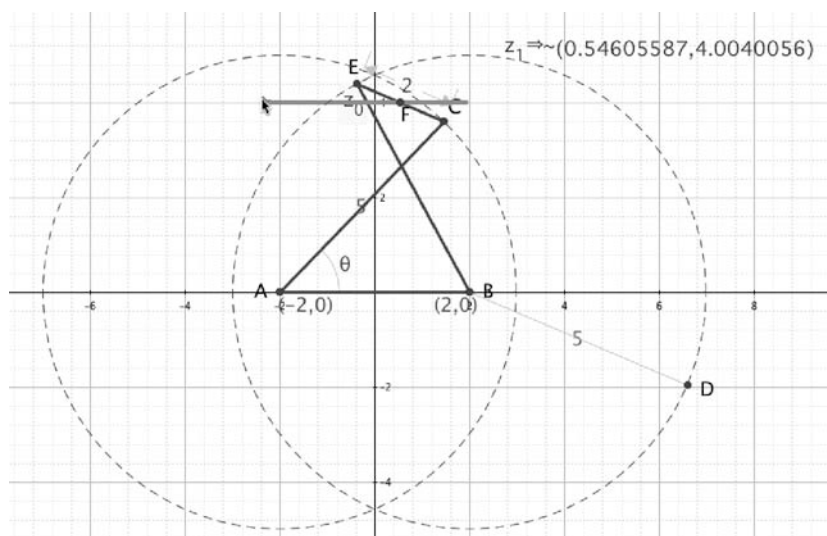


FIGURE 5. CHEBYSHEV'S LINKAGE.



algebra system is a very powerful feature that, as far as I know, is unique to GE.

## Analyzing A Gothic Arch

I found the following problem at the Cut-The-Knot [6] site.

The Gothic Arc ODC is constructed with two circular arcs of radius OD, one centered on O and one on D. The sequence of circles is constructed as shown in Figure 5 below so that each circle is tangent to the semicircle on diameter OD, the arc ODC, and the adjacent circles in the sequence. Show that the radius of each of these tangent circles is a rational fraction of OD. (This implies that the right triangles with one vertex at the center of a circle from the sequence, another at O, and the base on OB have rational sides. In particular, ODP is a 3–4–5 triangle.) [11]

To create the drawing shown in Figure 7, define circle O to have a variable radius of  $a$ , and then create circle E by defining it to be tangent to circle O with radius of  $a/2$ . Define Circle B as being tangent to circles E and O with radius  $a/2$ . Circle K is tangent to circles B, E and O. Circle P1 is tangent to circles B, O and K. Circle P2 is tangent to circles P1, B and O. Circle P3 is tangent to circles P2, B and O. In general, circle  $P_n$  is tangent to circles  $P_{n-1}$ , B and O.

Measuring OB, BP and OP shows that it is a 3–4–5 right triangle. Measuring the radii of the chain of circles P1, P2, ... gives you Table 1.

Circle	Radius
P1	$a/6$
P2	$a/11$
P3	$a/18$
P4	$a/27$

TABLE 1. RADII OF THE CHAIN OF CIRCLES PRODUCED IN FIGURE 5.

A little analysis of the sequence 6, 11, 18, 27 reveals that it is quadratic with an  $n$ th term of  $n^2 + 2n + 3$ . Therefore,

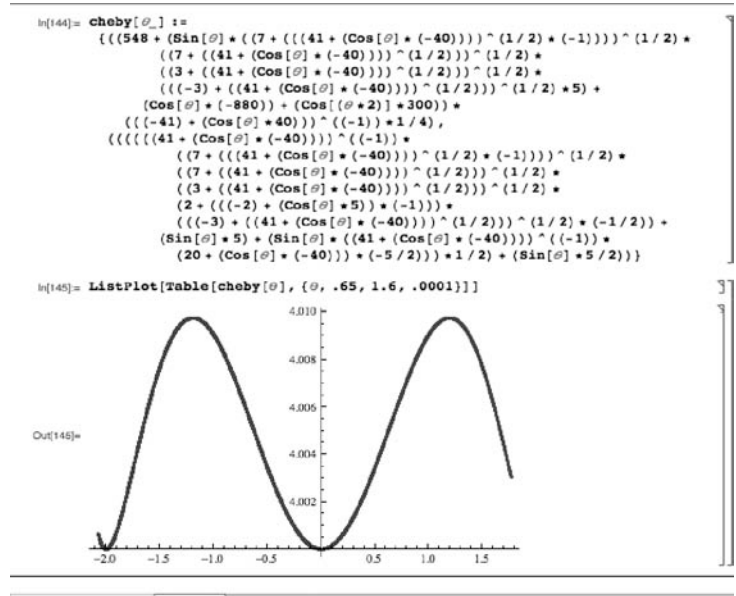


FIGURE 6. PLOT OF THE LOCUS OF POINT F USING MATHEMATICA.

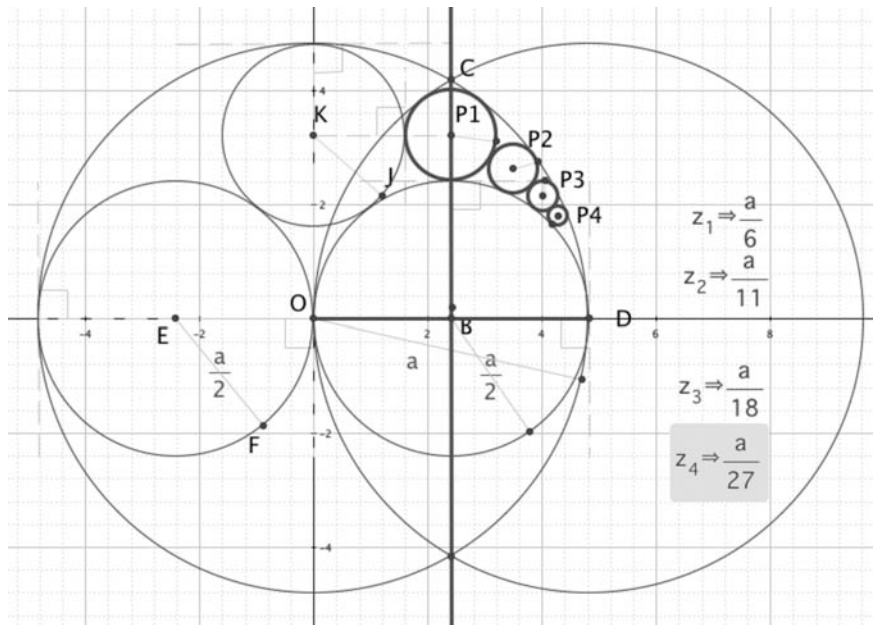


FIGURE 7. A GOTHIC ARC CONSTRUCTION.

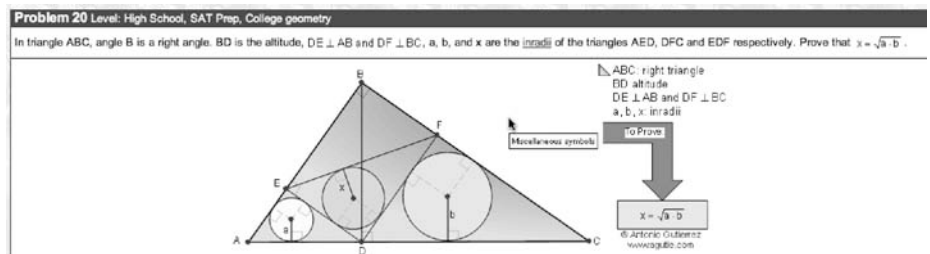


FIGURE 8. TYPICAL PROBLEM FROM GO GEOMETRY STEP-BY-STEP FROM THE LAND OF THE INCAS [5].

the  $n$ th circle  $P_n$  in the chain has radius  $a/n$ , where  $a = OD$ . A slightly different and more general analysis of this problem can be found at [11]. This problem is loaded with geometric gems.

### A Source of Interesting Problems

Here is my final problem. It is problem 20, as shown in **Figure 8**, from the Website Go Geometry Step-by-Step From the Land of the Incas [5]. It is presented in Figure 8 exactly as it appears at the site. The reasons I have include it are twofold. First, it is typical of the many interesting problems you can find at that site. Secondly, it is a problem for which it is very easy to create a diagram using *GE*.

*Geometry Expressions* is powerful, fun to learn, and it gave me an excuse to find interesting problems in old and new books and on the Internet. I plan to use it extensively this coming year to teach a senior seminar that will include an in-depth study of the conics from a geometric point of view. For those of you who are interested in learning more about *GE*, you can go to [2] and download a free 30-day trial copy. Also, the *GE* files used to produce this column can be found on my Website, [www.zebragraph.com](http://www.zebragraph.com).  $\square$

### References

- [1] More more information about the conference go to [http://www.exeter.edu/summer\\_programs/9402.aspx](http://www.exeter.edu/summer_programs/9402.aspx).
- [2] For information go to <http://www.geometryexpressions.com/>.
- [3] Bryant, J. and Sangwin, C., *How Round is Your Circle: Where Engineering and Mathematics Meet*, 2008, Princeton University Press, ISBN-13: 978-0-691-13118-4, <http://press.princeton.edu/titles/8624.html>.
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- [5] [http://www.gogeometry.com/recent\\_web\\_pages.html](http://www.gogeometry.com/recent_web_pages.html).
- [6] <http://www.cut-the-knot.org/index.shtml>.
- [7] <http://www.cut-the-knot.org/Curriculum/Geometry/EllipseInArbelos.shtml>.
- [8] <http://www.merriam-webster.com/dictionary/linkages>.
- [9] [http://en.wikipedia.org/wiki/Linkage\\_%28mechanical%29](http://en.wikipedia.org/wiki/Linkage_%28mechanical%29).
- [10] [3] pp. 28–30.
- [11] <http://www.cut-the-knot.org/Curriculum/Geometry/GothicArc.shtml>.

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Send solutions to old problems and any new ideas to the Geometer's Corner editor:  
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