

MORTGAGES AND SEA SHELLS: A DYNAMIC APPROACH

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This column focuses on iterating linear functions of both a real and a complex variable

and shows that there is some interesting geometry underlying the process. Here is a quick overview of how to iterate a linear function. (If you are unfamiliar with the process, [1] has a good introduction to iteration.) Start with a linear function in the form $f(x) = ax + b$ and a value x_0 and then calculate the sequence $x_0, x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)), x_3 = f(x_2) = f(f(f(x_0))), \dots$

The sequence $\{x_i\}$ is called the *orbit* of x_0 . For example let $f(x) = 0.5x + 4$ and $x_0 = 10$, then the orbit of 10 is $\{10, 9, 8.5, 8.25, \dots\}$. The first three are calculated:

$$x_1 = f(10) = 0.5(10) + 4 = 9$$

$$x_2 = f(f(10)) = 0.5(9) + 4 = 8.5$$

$$x_3 = f(f(f(10))) = 0.5(8.5) + 4 = 8.25$$

When iterating a function, we are interested in what happens to the orbit in the long run. In this example as n gets larger and larger x_n approaches 8. For this reason, 8 is called an *attracting fixed point*. Note that in this case $f(8) = 8$, which suggests that we could find the fixed point by solving the equation $f(x) = x$ or $x = 0.5x + 4$.

Here is another example. Let $f(x) = 2x + 1$. The orbit of 1 is $\{1, 3, 7, 15, 31, \dots\}$.

In this case, the orbit of 1 gets larger and larger and approaches infinity. Starting with a seed of -2 , you get an orbit of $\{-2, -3, -5, -9, -17, \dots\}$, which takes off towards negative infinity. There is a fixed point in this case. Solving $x = 2x + 1$ you learn that $x = -1$. This is an example of a *repelling fixed point* since the orbits of seeds greater than -1 go off to infinity and those less than -1 go off to negative infinity.

In general if you iterate, $f(x) = ax + b$ there will be a fixed point at $F = \frac{b}{1-a}$.

- F is attracting if $|a| < 1$;
- F is repelling if $|a| > 1$.

Any linear function with a fixed point can F be re-expressed as $f(x) = a(x - F) + F$. This is called *fixed point form*. Note that

$$\begin{aligned} f(f(x)) &= a(f(x) - F) + F \\ &= a[a(x - F) + F - F] + F \\ &= a^2(x - F) + F \end{aligned}$$

How about $f(f(f(x)))$?

$$\begin{aligned} f(f(f(x))) &= a\{a^2(x - F) + F\} - F + F \\ &= a\{a^2(x - F) + F\} - aF + F \\ &= a^3(x - F) + aF - aF + F \\ &= a^3(x - F) + F \end{aligned}$$

This suggests that if you iterated $f(x)$ n times you would get

$$f^{[n]}(x) = a^n(x - F) + F \quad (1),$$

a result that has a nice induction proof.

Equation (1) has an interesting geometric interpretation when you think of it in terms of vectors on a number line. In **Figure 1**, X , the seed, is where you start and F is the fixed point. The vector $X - F$ represents how far you are from the fixed point F . Clearly, $X - F + F = X$. Since $a(X - F)$ is a multiple of the vector $(X - F)$, $a(X - F)$ will either be longer or shorter than $(X - F)$ depending on the value of $|a|$. If $|a| > 1$ then $a(X - F)$ is longer than $(X - F)$ and $X_1 = a(X - F) + F$ is farther from the fixed point than X .

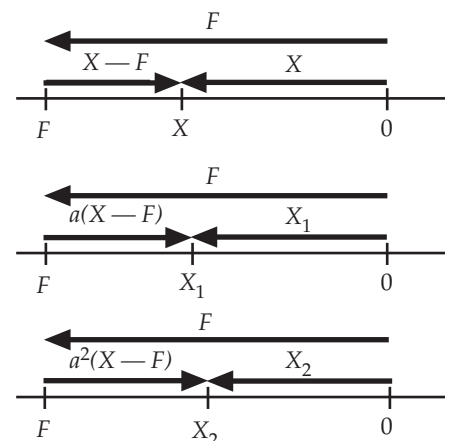


FIGURE 1. A VECTOR INTERPRETATION OF ITERATING $f(x) = ax + b$.

If $|a| < 1$ then $a(X - F)$ is shorter than $(X - F)$ and $X_1 = a(X - F) + F$ is closer to the fixed point.

Using a similar argument, $X_2 = f^{[2]}(X)$ is closer to the fixed point F than X_1 if $|a| < 1$ and farther away if $|a| > 1$.

Another approach is to look at the sequence $\{|X_1 - F|, |X_2 - F|, |X_3 - F|, \dots\}$. This is an increasing geometric sequence if $|a| > 1$ or a decreasing one if $|a| < 1$. With the preceding approach in mind, let's see how this applies to the world of finance.

Mortgages and Iterating Real Valued Linear Functions

Consider how a loan or mortgage works. The bank lends you an initial amount M . After one month, the bank calculates the interest, adds it to the balance and adds whatever payment P you made. If i is the interest rate per interest period then your balance after one month is $M + iM + P = (1 + i)M + P$. The bank continues to do this until your balance is \$0. Note that all the bank is doing to calculate your new balance is iterating the function $f(x) = (1 + i)x + P$. The sequence $\{f(M), f(f(M)), \dots\}$ gives your balance each month.

Here is an example. You borrow \$10,000 and are charged 0.5 % per month and make payments of \$500 per month. Your balance at the end of the first month is

$$-10000 + (.005)(-10000) + 500 = -9,550.$$

Notice that you have reduced the balance by an amount of \$450, which means you also paid the bank \$50 in interest.

Also notice that you could have rewritten the above as

$$(1.005)(-10000) + 500,$$

so all you are really doing is iterating the function

$$f(x) = 1.005x + 500.$$

This has a fixed point $-100,000$ and it is *repelling*. It also has an interesting interpretation. If you were to borrow \$100,000 at this interest rate and made payments of \$500 per month, all you would do is pay the interest and your balance would be $-\$100,000$ forever. This is the loan from Hell!

To see how long it would take you to pay off this loan we can use Formula 1. Substituting $1 + i$ for a and $\frac{-P}{i}$ for F , the balance $B(n)$ in the n th time period is

$$B(n) = (1 + i)(1 + i)^n \left(B(0) + \frac{P}{i} \right) - \frac{P}{i}$$

For this loan, $B(0) = -100000$, $B(n) = 0$, $i = 0.005$, and $P = 500$.

$$0 = 1.005^n(-10000 + 100000) - 100000$$

$$0 = 1.005^n(90000) - 100000$$

Using logs you get

$$n = \frac{\log\left(\frac{10000}{90000}\right)}{\log(1.005)} \approx 21.1247$$

So it takes a little more than 21 months to pay off the loan.

Think about the sequence defined by $1.005^n(90000)$. It is a geometric sequence that measures how far you are from the fixed point, and it grows slowly at first.

Does this explain the awful feeling you get when you start your mortgage payments and the balance hardly changes?

The Golden Spiral and Iterating the Linear Complex Function

$$f(z) = az + b$$

One of the highlights for me in teaching advanced algebra has always been teaching complex numbers in general and De Moivre's Theorem in particular, which states that if you multiply two complex numbers in polar form, $[r_1; t_1]$ and $[r_2; t_2]$, their product is equal to $[r_1 r_2; t_1 + t_2]$. In short, you multiply the lengths and add the angles.

Consider now the complex function $f(z) = az + b$ where z , a , and b are complex numbers. Expressed in polar form, $a = [r; t]$. This means that the function $f(z)$ rotates the complex number z through an angle of t , stretches it by a factor of r and then slides it b_2 units in the horizontal direction and b_1 units in the vertical direction. Just as in the case of the real valued linear function, f has a fixed point Φ . In this case, $\Phi = \frac{b}{1-a}$ and, as was the case with the real valued linear function $f(x) = ax + b$, $f(z)$ can be re-expressed as $f(z) = a(z - \Phi) + \Phi$. Using the fact that if α and β are complex numbers then $|\alpha\beta| = |\alpha||\beta|$, you can see that the sequence $\{|(z - \Phi)|, |\alpha||z - \Phi|, |\alpha^2||z - \Phi|, \dots\}$ again forms either an increasing or decreasing sequence depending on the length of a . Since we are now dealing in two dimensions, the sequence z_i either spirals in towards Φ or away from it. For example, **Figure 2** shows a sequence produced by iterating $az + b$ with $|a| > 1$. In both cases the sequence $\{z_i\}$ lies on a spiral.

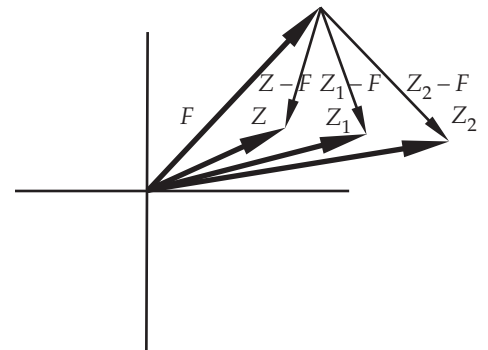
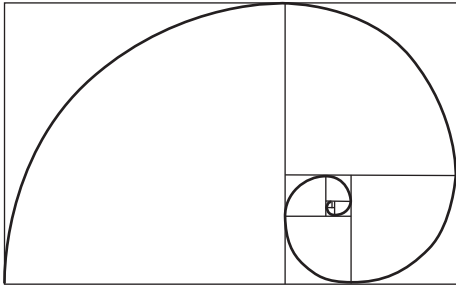
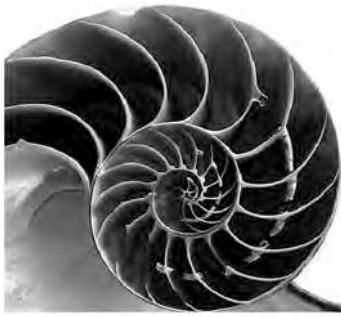


FIGURE 2. A VECTOR INTERPRETATION OF ITERATING $f(x) = ax + b$.

Let's put the preceding example to use. It is commonly thought that the spiral curve you find when you slice a seashell like the Chambered Nautilus in half is the same as the spiral known as the Golden Spiral shown in **Figure 3**. This spiral is formed by creating a series of nested Golden Rectangles and then creating a series of arcs inscribed in the squares. They look like they form a spiral: actually, they don't [2]. In what follows we will see why.



A) THE GOLDEN SPIRAL



B) CHAMBERED NAUTILUS

FIGURE 3.

Here's how to construct the Golden Spiral in Figure 3a using a geometric construction program.

1. Construct a Golden Rectangle. There are lots of ways to do this but here is a relatively easy one. Construct a segment AB of any length. Use the calculator to calculate the Golden Ratio, which we denote by ϕ and equals $\frac{1+\sqrt{5}}{2}$. Then use the dilate tool with a dilation factor equal to ϕ to create a point C on AB such that AC has length equal to that of AB times ϕ . Construct a rectangle with length equal to that of AC and width equal to that of AB .
2. Label the rectangle $WXYZ$ as shown in **Figure 4**.
3. With W as center rotate rectangle $WXYZ$ through -90° creating rectangle $WX'Y'Z'$.
4. With W as center dilate rectangle $WX'Y'Z'$ by a scale factor of $\frac{1}{\phi}$ creating rectangle $WX''Y''Z''$ as shown in Figure 4.

5. Create a translation equal to vector $X''Y$ and translate rectangle $WX''Y''Z''$. The result $W'YZX'$ shown in Figure 4.
6. Repeat steps 3–6 on rectangle $W'YZX'$ and on all resulting images.

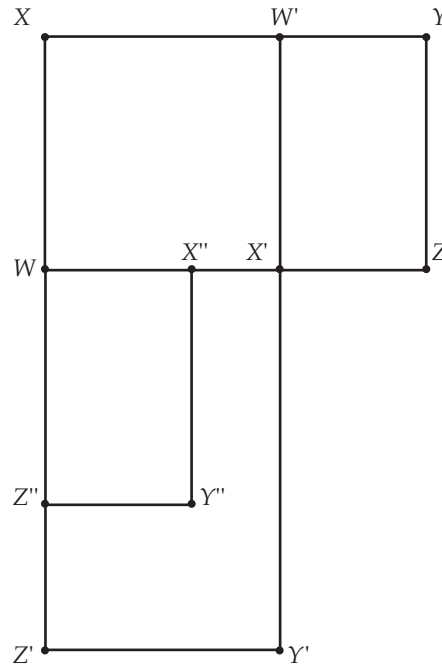


FIGURE 4. FIRST STEP IN CONSTRUCTING THE GOLDEN SPIRAL.

Continue this process as long as you can, construct the appropriate circular arcs in each square and you will create the spiral shown in Figure 3a.

This certainly looks like a spiral but it is a fraud since it is composed of circular arcs. Spirals come in two varieties: Archimedean and Logarithmic. The difference between the two types is that if you draw a ray from the center O of the spiral and mark the successive points P_i , where a ray from the center of the spiral intersects the spiral, then the spiral is:

- Archimedean if $OP_{i+1} - OP_i =$ a constant
- Logarithmic if $\frac{OP_{i+1}}{OP_i} =$ a constant.

Looking carefully at the Golden Spiral and the spiral in the Nautilus Papilius, you can see that both spirals are Logarithmic and not Archimedean.

For the spiral to be a Golden Spiral, the ratio has to equal ϕ . Here is how to use iteration of a linear complex function to create a Golden Spiral with the same method used to produce the sequence of Golden Rectangles in Figure 4.

Assume that rectangle $WXYZ$ is centered at the origin W and that point X has complex coordinates 1. The complex linear function

$$f(z) = \left(\frac{-i}{\phi}\right)z + 1 + i$$

rotates z by -90° , shrinks it by a factor of $\frac{1}{\phi}$ and slides the result 1 unit to the right and up 1. By creating the orbit of 0 we can create a set of points that lie on the Golden Spiral. Again using the work we did earlier, we can re-express $f(z)$ in the form $f(z) = a^n(z - \Phi) + \Phi$ as follows:

$$f^{[n]}(z) = \left(\frac{-i}{\phi}\right)^n (z - F) + F,$$

$$\text{where } F = \left(\frac{1+i}{1+\frac{i}{\phi}}\right)$$

Since we know that 0 is on the spiral, we can let $z = 0$ and plot the first 10 points in the orbit of 0 by calculating and then plotting the complex sequence

$$\{0, f[1](0), f[2](0), \dots, f[9](0)\}.$$

Figure 5 shows a plot of the first 10 points in the orbit of 0 joined by line segments.

This gives only 10 points on the spiral so there is more work to do. We can use DeMoivre's Theorem again. The function $f(x) = \left(\frac{-i}{\phi}\right)^x$ is defined for all

real x . If we express $a = \frac{-i}{\phi}$ in polar form as

$$a = \left(\frac{1}{\phi}\right) \left(\cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \right),$$

$$\text{then } a^x = \left(\frac{1}{\phi}\right)^x \left(\cos\left(\frac{-\pi}{2}x\right) + i \sin\left(\frac{-\pi}{2}x\right) \right).$$

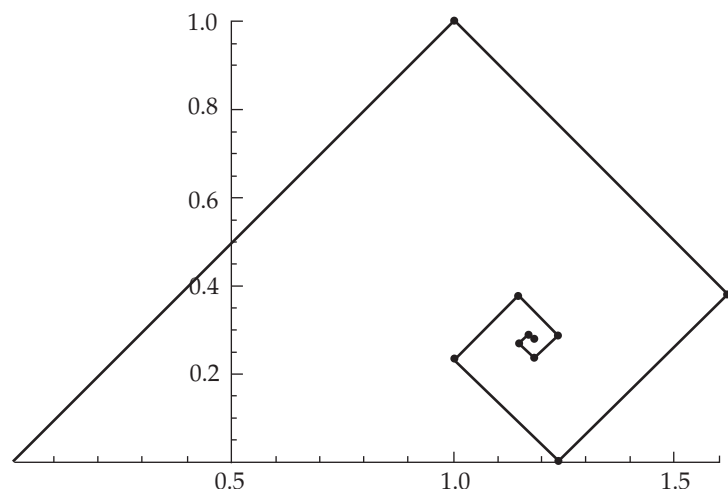


FIGURE 5. THE FIRST 10 POINTS IN THE ORBIT OF 0 JOINED BY LINE SEGMENTS.

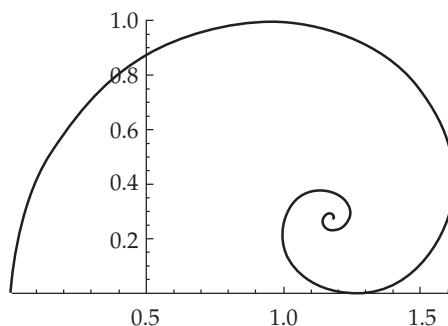


FIGURE 6. A) A GOLDEN SPIRAL

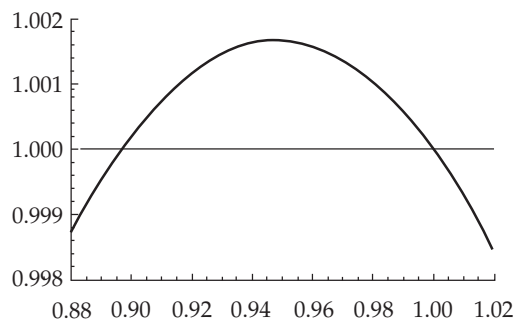


FIGURE 6. B) AN ENLARGEMENT OF 6A NEAR (1, 1).

Using Mathematica we can get the plot of $f(z)$ for $0 < x < 20$ in steps of 0.005 shown in **Figure 6a** [4]. There is a problem if we were to overlay this graph on the Golden Rectangle shown in Figure 3a. If we blow up the graph around the point (1, 1) as shown in **Figure 6b**, we see that the spiral actually goes outside the Golden Rectangle.

This means that the Golden Spiral doesn't fit inside the Rectangle and shows that one cannot fit an Archimedean spiral inside the Golden Rectangle.

There is a second reason the Golden Spiral doesn't fit spirals on shells like the Chambered Nautilus. Since this is

an Archimedean Spiral, the radius increases at a constant rate per revolution, which in the case if the Golden Spiral equals $\phi^4 \approx 6.85$. This means that the radius of the spiral increases by a factor of more than 6 per revolution.

If you were to measure how the spiral on a Chambered Nautilus expands like the one shown in Figure 3, you would find that it expands at a rate closer to 1.3 per revolution [2]. This means that the Golden Spiral fails on two counts to be a good match for the spiral found in a Chambered Nautilus.

It is interesting that the method we used to calculate the size of an affordable 30-year mortgage is very

similar to the method we used to study the curve thought by many to best fit the spiral found in many sea shells. Such is the beauty of mathematics. \square

References

- [1] Picciotto, Henri, and Jonathan Choate. 1997. Iterating linear functions: an introduction to dynamical systems. *Mathematics Teacher* 90(2): 122–136. This article is available at <http://www.picciotto.org/math-ed/iterating/mt/ss.pdf>.
- [2] Sea Shell Spirals can be found at Ivars Peterson's MathTrek: http://www.maa.org/mathland/mathtrek_04_04_05.html.
- [3] Sharp, J. 2002. Spirals and the golden section. *Nexus Network Journal* 4(1). Available at http://www.nexusjournal.com/Sharp_v4n1-intro.html.
- [4] Copies of the Mathematica notebook used to create the spirals in Figures 5 and 6a can be found at <http://www.zebragraph.com>.



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