Text

JONATHAN CHOATE

A HANDS-ON APPROACH TO FINDING PLATONIC VOLUMES

Ross Honsberger states in *Mathematical Gems II*:

> Solid geometry is often a more complicated subject than plane geometry, for it is undoubtedly more difficult for the mind's eye to establish and maintain a constant picture of the relevant positions of objects in three-dimensional space. Flat figures are much easier to think about and to describe to others. Thus the pursuit of solid geometry demands greater motivation.

(Honsberger, R., Mathematical Gems II, MAA, 1976, p. 90 ISBN 0883853027)

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.

In my last column, I showed how to find the volume of anti-prisms. After finishing it, it occurred to me that you never see in any textbooks mention of how to find the volumes of icosahedra and dodecahedra. This Geometer's Corner is about how you can use a set of manipulatives, Zometools, to build models of each that suggest a way to find their volumes and surface areas. We will be doing a lot of work with regular pyramids, pyramids with bases that are regular polygons and lateral faces that are isosceles triangles. **Figure 1** shows a generic regular pyramid. **Box 1** displays formulas for finding the various measurements related to the pyramid. I have created a spreadsheet that does all the calculations and will use it later in the article.

Figure 1 depicts a regular pyramid $PB_1 B_2 \ldots B_n$, with regular polygonal base $B_1 B_2 \ldots B_n$ with edges b units long and lateral edges s units long. OP is the height, Q is the midpoint of side B_1B_2, O is the center of the base, and R is the foot of the altitude from B_1 to edge B_2P. $\angle OQP$ is the dihedral angle formed by the base and a side, and $\angle B_1RB_3$ is the dihedral angle formed by two adjacent lateral faces. Using right angle trig and the Law of Cosines you can derive the formulas in Box 1.

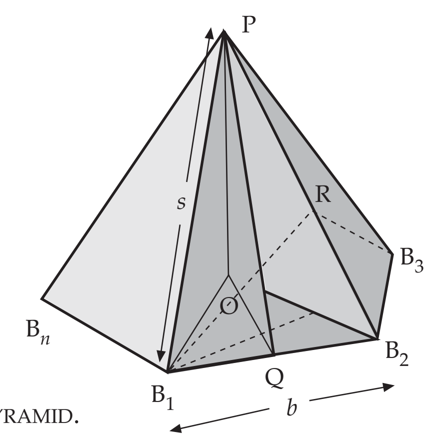


FIGURE 1. A GENERIC REGULAR PYRAMID.

- $OQ = \dfrac{b}{2\tan\left(\frac{180}{n}\right)}$
- Area $PB_1B_2 \ldots B_n = n\left(\frac{b}{2}OQ\right)$
- $OB_1 = \dfrac{b}{2\sin\left(\frac{180}{n}\right)}$
- $OP = \sqrt{B_1P^2 - OB_1^{\,2}}$
- Volume $PB_1B_2 \ldots B_n =$ $n\left(\frac{1}{3}\right)(\textit{area}\ PB_1B_2 \ldots B_n)OP$
- Base-Face Dihedral Angle $OQP =$ $\tan^{-1}\left(\frac{OP}{OM}\right)$
- $\angle B_1PB_2 = \sin^{-1}\left(\frac{b}{2s}\right)$
- $\angle B_1B_2P = 90 - \dfrac{\angle B_1PB_2}{2}$
- $B_1R = b\sin(\angle PB_2B_1)$
- $B_3R = B_1R = r$
- $\angle B_1B_2B_3 = 180 - \dfrac{360}{n}$
- $B_1B_3 = \sqrt{2b^2 - 2b^2\cos(\angle B_1B_2B_3)}$
- Face-Face Dihedral angle $\angle B_1RB_3 = \cos^{-1}\left(\dfrac{2r^2 - (B_1B_3)^2}{2r^2}\right)$

BOX 1. VOLUME, AREA AND ANGLE FORMULAS FOR A REGULAR PYRAMID.

For a good development of these formulas visit http://mathworld.wolfram.com/RegularPyramid.html

Playing with Zometools

Zometools consist of hubs and struts. The hubs are spheres with rectangular, triangular and pentagonal holes in them as shown in **Figure 2.**

FIGURE 2. ZOMETOOL HUB.

The Struts come in a variety of lengths and colors. They consist of

- Three different lengths of blue struts each ending in a rectangular peg.
- Three different lengths of red struts each ending in a pentagonal peg.
- Three different lengths of yellow struts each ending in a triangular peg.

The lengths of the struts are given below, assuming that the short blue strut is one unit long. Φ is the Golden Ratio, which equals $\frac{1+\sqrt{5}}{2}$.

	Short	Medium	Long
Blue	1	Φ	Φ^2
Red	cos(18°)	cos(18°)Φ	cos(18°)Φ^2
Yellow	cos(30°)	cos(30°)Φ	cos(30°)Φ^2

TABLE 1. LENGTHS OF ZOMETOOL STRUTS.

The struts and hubs are well made and durable and just invite you to play with them. With this in mind, I took one of the hubs and filled all the pentagonal holes with short red struts and then placed hubs on the end of each of the 12 struts producing what I like to call the Little Red Star Burst. As I looked at it, it became clear that I might be able to connect each of the hubs with its nearest neighbor. I did this with short blue struts and got the icosahedron shown in **Figure 3.**

As I looked at the model I noticed that the icosahedron consisted of 20 triangular pyramids with vertices at the center of the Star Burst, triangular bases made up of short blue struts with length 1 unit, and lateral short red edges that are cos(18°) units in length. Therefore to find the volume of the icosahedron all I needed to do was to find the volume of one of the pyramids shown in **Figure 4** and multiply by 20. To do this, I used a spreadsheet that evaluated the formulas in Box 1 with $b = 1$, $s = \cos(18°)$ and $n = 3$. The results are shown in **Table 2.**

Inspired by what I found with the Little Red Star Burst, I started to build a Little Yellow Starburst and found quickly that joining neighboring hubs with short red struts didn't work as nicely as it did in the preceding case. So I built a Medium Yellow Star Burst and found to my delight that adjoining hubs could again be joined with short blue struts. When I finished connecting all the neighboring hubs I ended up with the dodecahedron shown in **Figure 5.** As was the case with the icosahedron, the dodecahedron can be dissected into congruent regular pyramids (**Figure 6**) that in this case have a pentagonal base with edges 1 unit in length and medium yellow lateral edges that are cos(30°)Φ in length. Using my spreadsheet again with $b = 1$, $s = \cos(30°)$, $n = 5$, and 12 faces, I got the results in **Table 3.**

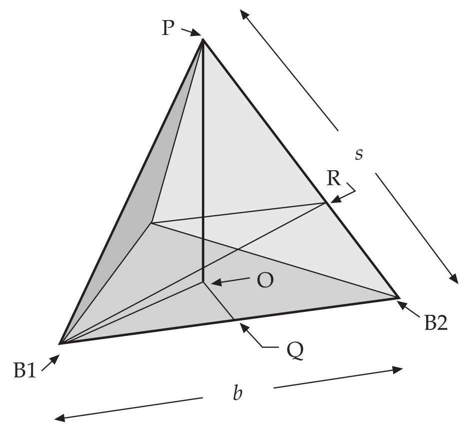


FIGURE 4. ICOSAHEDRAL PYRAMID.

s	b	n	# of faces	Phi
0.95105652	1	3	20	1.61803399

OQ	0.28867513	<OQP	69.0948426
Area Base	0.4330127	<B1PB2	31.7174744
OB1	0.57735027	<B1B2P	58.2825256
OP	0.75576131	B1R	0.85065081
Volume	0.10908475	B3R	0.85065081
		<B1B2B3	60
		B1B3	1
		<B1RB3	72

Total Surface Area	8.66025404
Total Volume	2.18169499
Polyhedral Dihedral Angle	138.189685

TABLE 2. REGULAR PYRAMID DATA FOR AN ICOSAHEDRON.

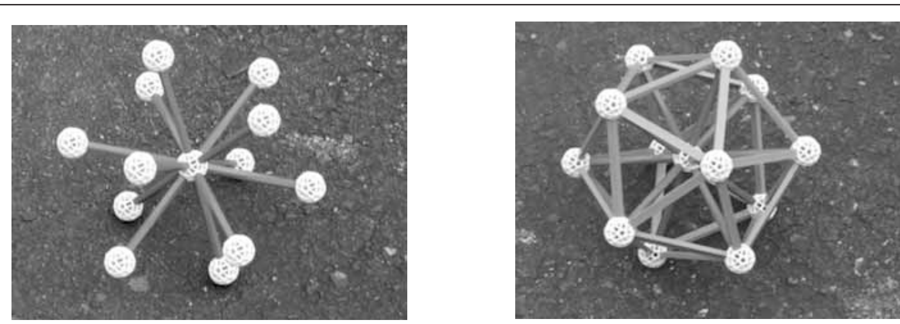


FIGURE 3. LITTLE RED STAR BURST AND RESULTING ICOSAHEDRON.

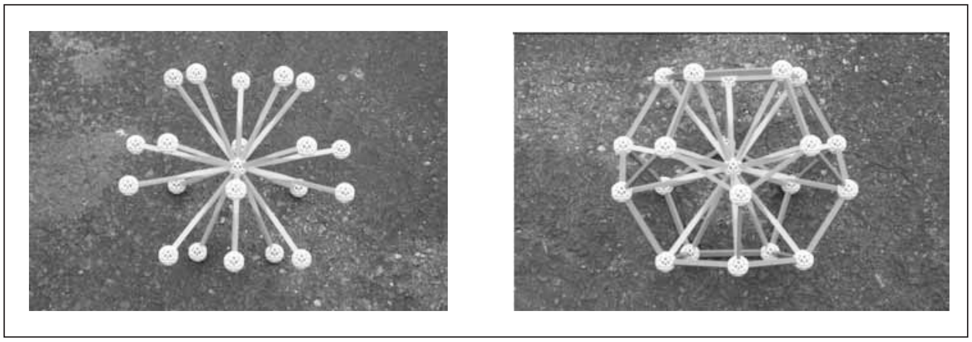


FIGURE 5. MEDIUM YELLOW STAR BURST AND RESULTING DODECAHEDRON.

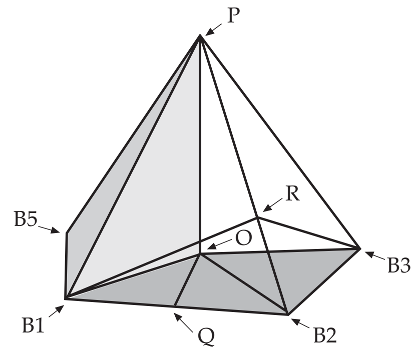


FIGURE 6. DODECAHEDRAL PYRAMID.

s	b	n	# of faces	Phi
1.40125854	1	5	12	1.61803399

OQ	0.68819096	<OQP	58.2825256
Area Base	1.7204774	<B1PB2	20.9051574
OB1	0.85065081	<B1B2P	69.0948426
OP	1.11351636	B1R	0.93417236
Volume	0.63859325	B3R	0.93417236
		<B1B2B3	108
		B1B3	1.61803399
		<B1RB3	120

Total Surface Area	20.6457288
Total Volume	7.66311896
Polyhedral Dihedral Angle	116.565051

TABLE 3. REGULAR PYRAMID DATA FOR A DODECAHEDRON.

I was on a roll now so the next thing I built was a Short Blue Star Burst. Again, the short reds didn't show much but the medium blues did. I could join each hub to its nearest neighbor with a short blue creating a polyhedron where each vertex is surrounded by two equilateral triangles and two pentagons. When I finished I had a semi-regular polyhedron known as an icosadodecahedron **Figure 7.**

As I had done previously I decided to calculate the volume. The icosdodecahedron is made up of triangular and pentagonal pyramids, but I needed to figure out how many of each. I could have counted them but decided to do a little calculation instead. Doing a little counting I found that there are 30 vertices, and at each vertex two triangles and two pentagons met. Thanks to some communications with Jeff Weeks, I could calculate the number of triangular faces by noticing that that there are two triangles at reach vertex, but each triangle has three vertices so the number of triangles must equal $\frac{2\cdot 30}{3} = 20$. Similarly, there are $\frac{2\cdot 30}{5} = 12$ pentagons. I'll leave it to the interested reader to show that:

- A right triangular pyramid with base length 1 and lateral edges of length Φ has volume 0.2182 units^3 and dihedral angle 79.188º.
- A right pentagonal pyramid with base length 1 and lateral edges of length Φ has volume 0.7893 units^3 and dihedral angle 63.435º.
- Therefore, an icosadodecahedron with sides of length l has volume 13.8355 units^3 and dihedral angle 142.623º.

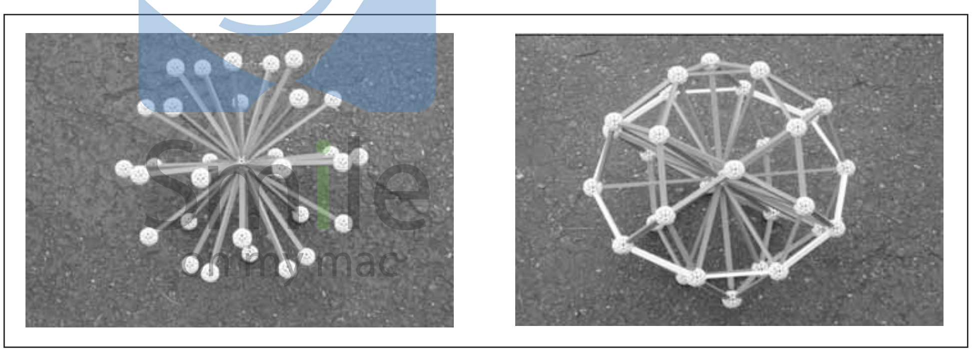


FIGURE 7. MEDIUM BLUE STAR BURST AND RESULTING ICOSADODECAHEDRON.

More on Strut Lengths

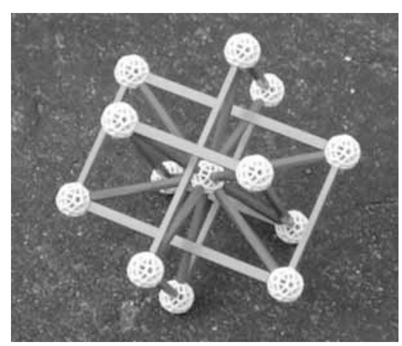


FIGURE 8. THREE GOLDEN RECTANGLES IN AN ICOSAHEDRON.

Using the icosahedron I built you can show that each vertex lies on one of three rectangles as shown in **Figure 8.** Each rectangle has width short blue, length medium blue and diagonal two short reds long. Let R = length of short red, 1 = length of short blue, and Φ = length of medium blue. Since we have a rectangle, we have

$$4R^2 = 1 + \Phi^2$$

Since $1 + \Phi = \Phi^2$, (showing this using the definition of Φ is a good exercise.)

$$4R^2 = 2 + \Phi$$

$$R = \frac{\sqrt{2+\Phi}}{2}$$

$$R = \frac{\sqrt{2+\left(\frac{1+\sqrt{5}}{2}\right)}}{2} = \frac{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}}{2} =$$

$$\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4} = \cos(18°)$$

I used a TI-89 calculator to show that $\cos(18°) = \frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4}$. You could also show it by using a Golden Triangle ABC, an isosceles triangle with equal sides AB and AC Φ units long and base BC 1 unit long. To show the result let M be the midpoint of BC and show that

i) $\angle ABC = 36^o$

ii) $\angle ABM = 18^o$

iii) $\cos(\angle ABM) = \frac{AM}{AB}$

iv) $\frac{AM}{AB} = \frac{\sqrt{\Phi^2 - \frac{1}{4}}}{\Phi}$ AM

v) $\frac{\sqrt{\Phi^2 - \frac{1}{4}}}{\Phi} = \frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4}$

Using the dodecahedron I built you can show that a cube medium blue on a side can be inscribed inside the dodecahedron as shown in **Figure 9**. The cube has diagonals that are two medium yellows in length as shown in **Figure 10**. Since a medium yellow has length ΦY where Y is the length of a short yellow, we have a right triangle with hypotenuse $2\Phi Y$ units long and legs with lengths Φ and $\sqrt{2}\,\Phi$. Which gives us the following.

$$4\Phi^2 Y^2 = \Phi^2 + 2\Phi^2$$

$$4Y^2 = 3$$

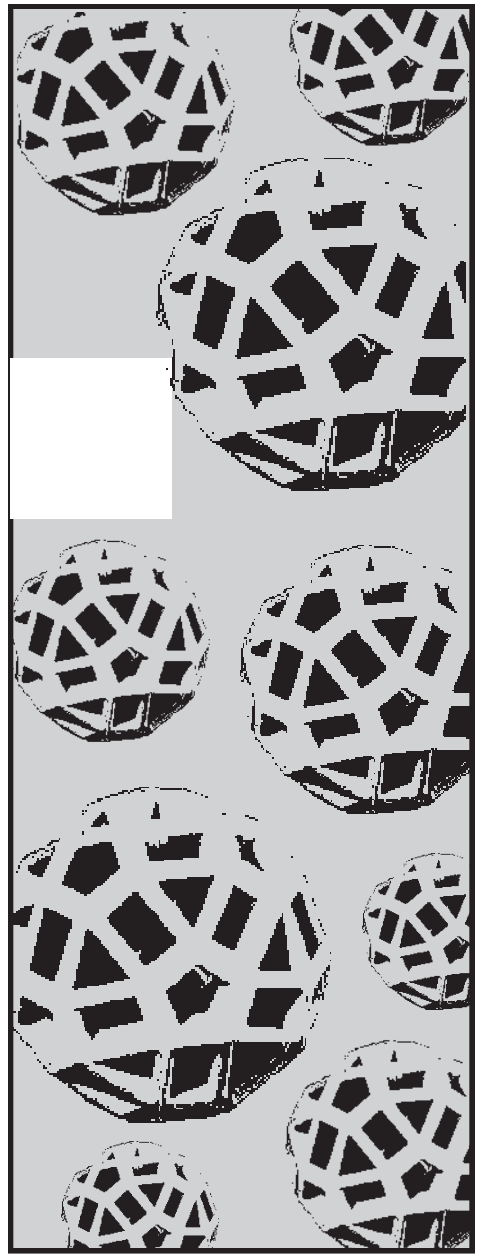
$$Y = \frac{\sqrt{3}}{2} = \cos(30°)$$

The preceding just scratches the surface of what you can do with Zometools. I have said nothing about what you can do with the green struts, but it should come as no surprise that you can build polyhedra that are cube and octahedron related. The important point is that they provide a very rich environment in which students and teachers can learn a lot about three dimensional shapes.

I think I have had more fun putting this article together than any previous Geometer's Corner. I learned a lot just by playing with the Zometools. The material about using Zometools to find the volumes of icosahedra and doceahedra I have used in class. The material on the icosadodecahedron is new, and I can't wait to use it in my geometry class.

I am more convinced now than ever that manipulatives like Zometools can be used to learn three-dimensional geometry in new and exciting ways because they make it much easier to describe and think about three-dimensional objects.

For more information about Zometools visit http://www.zometool.com/ ❑



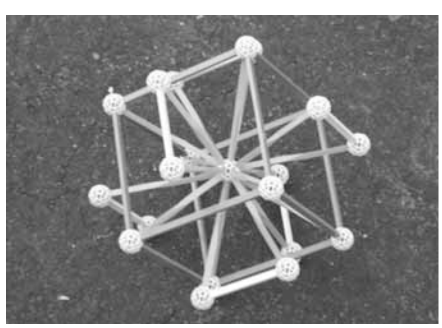


FIGURE 9. CUBE INSCRIBED IN A DODECAHEDRON.

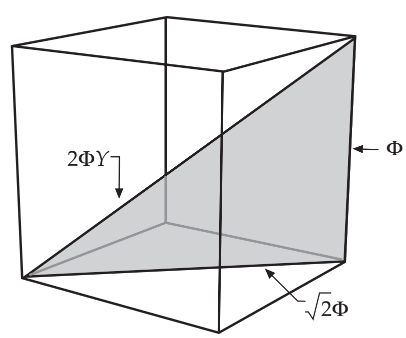


FIGURE 10. CUBE INSCRIBED IN A DODECAHEDRON.