Elegant Mathematics

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Although the role of solid geometry in the secondary mathematics curriculum is sometimes minimal, many contemporary geometry books still include material on pyramids, prisms, and anti-prisms. Only occasionally is some mention made of the five Platonic Solids, which is unfortunate because they are the source of some elegant mathematics. This edition's Geometer's Corner will focus on polyhedra and how they can be studied making use of both elementary algebra and a new easy to use computer-aided design tool called "SketchUp."

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.

Polyhedra are three-dimensional objects that consist of vertices, edges, and faces. A polyhedron is said to be regular if all the faces are identical regular polygons and the same number of faces meets at each vertex. In what follows, we will also assume that all the polyhedra we study are convex and topologically equivalent to a sphere. There are only five possible regular polyhedra since the only possible configurations at each vertex are three, four, or five equilateral triangles; three squares; or three pentagons. The five possibilities are shown in **Figure 1.** Beneath each polyhedron is what is known as a net. Nets can be cut out, folded, and taped together to form polyhedra.

In order to figure out how many vertices, edges, and faces each Platonic Solid has, most textbooks rely on the building of a model using cardboard polygons or commercial manipulatives such as Polydrons, Jovos, Geofix, or Zometools. Here is another way of doing this, and it involves some algebra and two very useful theorems, one due to Descartes and the other to Euler.

A polyhedron is made up of edges, vertices, and faces. For example a cube has eight vertices, twelve edges, and six faces. If a polyhedron has V vertices, E edges, and F faces, Euler showed in 1752 that there is an elegant relationship between V, E, and F. He proved Euler's Theorem.

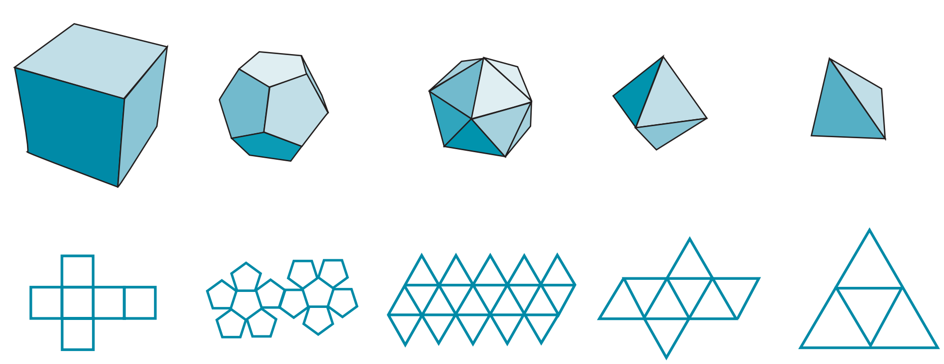


Figure 1. The five Platonic Solids. [1]

Euler's Theorem: In any polyhedron with V vertices, E edges, and F faces,

$$V - E + F = 2. \qquad \text{(Eq. 1)}$$

At the same time, Euler published another interesting result about angles in a polyhedron. (Descartes actually discovered it in 1620 but never published it.) The takeout angle (TOA) of a vertex is found by summing the angles of the polygons that meet at the vertex and subtracting that amount from 360 degrees. This is also what you get if you were to unfold the polyhedron and lay the vertex flat. The takeout angle is the resulting gap. Descartes showed that if you sum all the takeout angles for all the vertices in a polyhedron you will always get 720 degrees.

Descartes' Theorem: The total takeout of any polyhedron is 720 degrees.

Here are some other useful relationships that hold for any polyhedron. Take any polyhedron, go to each vertex and count the edges that meet at that vertex. Sum them all, and you always get twice the number of edges since each edge is double counted because each edge joins two vertices. The degree of a vertex is the number of edges that meet at that vertex. If you sum the degrees of all the vertices you always get twice the number of edges. In the case of a regular polyhedron that has V vertices and n edges meeting at each vertex

$$nV = 2E. \qquad \text{(Eq. 2)}$$

Now go to each face and calculate how many edges bound that face. Sum what you got for all of the faces, and you always get twice the number of edges since each edge is double counted because each edge belongs to two faces. In the case of a regular polyhedron that has F vertices and p edges bounding each face

$$pF = 2E. \qquad \text{(Eq. 3)}$$

We now have three equations that can be used to derive more information about the Platonic Solids. For example, consider the Platonic Solid that has five equilateral triangles meeting at each vertex. In this case, $n = 5$ and $p = 3$. The takeout angle for this solid is $360 - 5 * 60 = 60$. The total takeout is equal to $60V$ so you have $60V = 720$. Solve this for V, and you get $V = 12$.

Using Equation 2 with $n = 5$, you get

$5V = 2E$

$5(12) = 2E$

So $E = 30$.

To find F use Equation 1 with $V = 12$, $E = 30$ and you can show that $F = 20$. This solid is known as the icosahedron (*icosa* from the Greek word for 20).[2] It has twelve vertices, thirty edges, and twenty faces. I leave it to you to show that for an icosahedron $3F = 2E$ and that you will get the same values for V, E, and F by solving the system of equations

$5V = 2E$

$3F = 2E$

$V - E + F = 2.$

This set of equations is often referred to as the defining equations for an icosahedron.

Table 1 shows some very interesting relationships. For example:

- The number of vertices in an octahedron is equal to the number of faces in a cube, and the number of vertices in a cube is equal to the number of faces in an octahedron. Because of this the cube and the octahedron are said to be duals of each other.

- The number of vertices in an icosahedron is equal to the number of faces in a dodecahedron, and the number of vertices in a dodecahedron is equal to the number of faces in an icosahedron. The icosahedron and the dodecahedron are duals of each other.

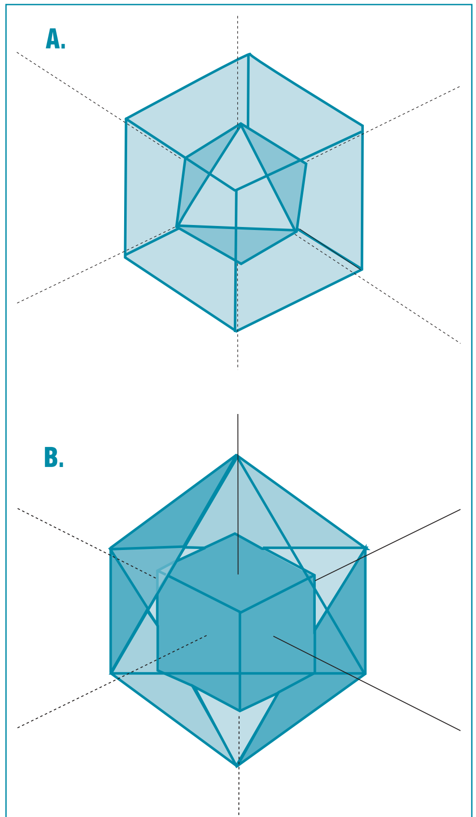


FIGURE 2. A) OCTAHEDRON AS DUAL OF A CUBE B) CUBE AS DUAL OF OCTAHEDRON

Name	n	p	V	E	F	TOA
Tetrahedron	3	3	4	6	4	180
Cube	**3**	**4**	**8**	**12**	**6**	**90**
Octahedron	**4**	**3**	**6**	**12**	**8**	**120**
Icosahedron	*5*	*3*	*12*	*32*	*20*	*60*
Dodecahedron	*3*	*5*	*20*	*32*	*12*	*36*

TABLE 1.

Now consider duality. To visualize the dual of a cube do the following: in your mind visualize a cube and the centers of all its faces; now imagine that any two centers that belong to faces that share an edge are joined by a line segment. These segments form an octahedron.

Repeat the above starting with an octahedron, and you get a cube.

Figures 2A and **2B** show this duality.

Repeat the above with a dodecahedron and you get an icosahedron.

Repeat the above with an icosahedron and you get a dodecahedron.

One can use a similar method of algebraic analysis to derive information about the Archimedian, or semi-regular, polyhedra.[3] These are polyhedra whose faces consist of two or more regular polygons arranged in such a way that each vertex has the same configuration of faces at each vertex. For example, consider the polyhedron whose faces are squares and equilateral triangles arranged in such a way that at each vertex there is a square, a triangle, a second square, and a second triangle in that order. Using an algebraic method similar to the one shown above one can deduce not only how many vertices, edges, and faces the polyhedra has, but also how many squares and triangles there are altogether. This method requires four steps.

Step 1. Calculate the takeout angle, and use it and Descartes' Theorem to calculate how many vertices the polyhedron has. In this case, the takeout angle is equal to 360 – (90 + 60 + 90 + 60) = 60 degrees. Descartes' Theorem tells us that

$60V = 720$

and therefore $V = 12$.

Step 2. In order to find the number of edges, we can use the equation $4V = 2E$ so

$4(12) = 2E$

$E = 24$

since the degree of each vertex is 4.

Step 3. To find F, we can use Euler's Theorem $V - E + F = 2$. In this case,

$12 - 24 + F = 2$

$F = 14$.

Step 4. Let S equal the number of squares and T the number of triangles. Since each square has 4 edges, the polyhedron has $4S$ edges that belong to the squares. Similarly, there are $3T$ edges that belong to the triangles. Adding these we get $4S + 3T$, which must equal twice the number of edges because each edge belongs to both a square and a triangle. Therefore, we have

$4S + 3T = 2E = 48$.

Since we already know that there are 14 faces we also have

$S + T = 14$.

Solving this system of equations below gives us $S = 6$ and $T = 8$.

$4S + 3T = 48$

$S + T = 14$.

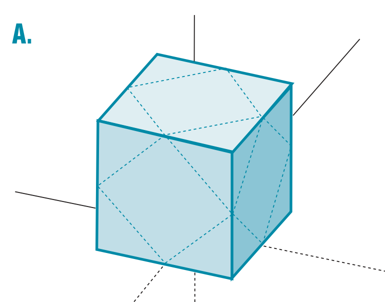
Another way of doing this uses ratios. It can be shown that the ratio of the total number of triangular edges to the total number of square edges, $3T{:}4S$, should equal the ratio of the number of triangles to the number of squares at each vertex, 1:1. This gives us $3T{:}4S$ = 1:1 or $3T = 4S$. This combined with $S + F = 14$ gives us another set of two equations.

For good reasons this polyhedron is called a cuboctahedron. To see why this name makes sense note that the number of triangular faces is equal to 8, which is the number of vertices in a cube and the number of faces in an octahedron, and that the number of square faces is equal to 6, which is the number of faces in a cube and the number of vertices in an octahedron. Notice the duality here. The cubeoctahedron is directly related to the cube and its dual the octahedron.

Here is how to build a cuboctahedron. Start with a solid cube and mark the midpoint of each edge and then join any two midpoints that belong to the same face (see **Figure 3A**).

Note that each vertex of the cube is also the vertex of a triangular pyramid. If each of these pyramids is cut off the cube, you get a solid like the one shown in **Figure 3B**. This is a cuboctahedron, and this process is known as vertex truncation. What do you think happens if you truncate the vertices of an octahedron? You again

A.



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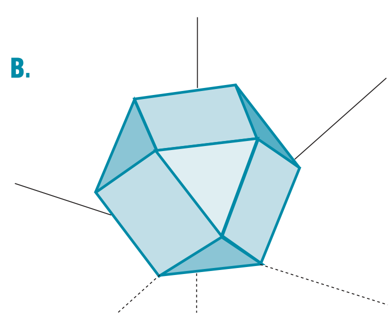


FIGURE 3. A) CUBE WITH MIDPOINTS OF SIDES JOINED. B) CUBE WITH VERTICES TRUNCATED.

get a cuboctahedron. This illustrates a major theorem in polyhedra theory that says that if you truncate two polyhedra that are duals you end up with the same polyhedron. It should come as no surprise that the polyhedra you create by truncating an icosahedron and a dodecahedron are identical and are called icosadodecahedra.

I recently discovered a piece of software that allows you to play with polyhedra in new and exciting ways. It is the software that I used to create most of the figures for this article. The software is "SketchUp," an architectural tool developed by the *AtLast Software*. The software has received rave reviews in the architectural world and may be of great use in the math classroom because it allows you to play with polyhedra and other 3D objects. A demo copy, which is good for 8 hours, is free, available for both Macs and PCs, and can be downloaded from www.sketchup.com. What follows is a brief description of the software and how I used it to create a cuboctahedron. When the program opens, you will get a set of 3D axes and a toolbar. To create a cuboctahedron I did the following.

1. Use the polygon tool to create a square. See **Step 1.**

2. Use the extrusion tool to create a cube. The extrusion tool pulls the square from the original surface in a perpendicular direction. See **Step 2.**

3. Pick a vertex and use the line tool to join the midpoints of the edges meeting at that vertex. See **Step 3.**

4. Use the erase tool to erase the line segments just formed. The line segments disappear leaving a triangular face. Use the paint tool to color this face. See **Step 4.**

5. Repeat the preceeding on the other seven vertices and you have a cuboctahedron. To do this you have to use the flyaround tool, which allows you to use a mouse to move around the solid. See **Step 5.**

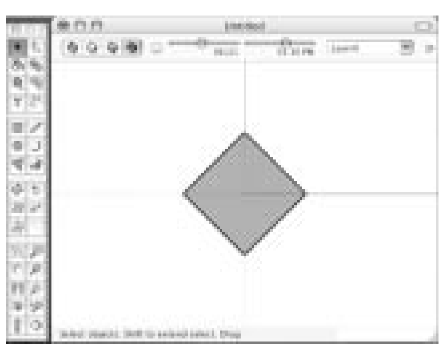
Once you get the hang of it, you can do some very interesting operations. In the tool bar there are zoom tools, a scaling tool, a rotational tool, a tape measure tool for measuring lengths, a protractor for measuring angles, and a variety of tools for viewing your objects from different positions-—just to mention a few. If you teach geometry or are looking for easy-to-use software for creating diagrams with three-dimensional objects, I urge you to check this out.

To learn more about polyhedra theory and how it can be integrated into the secondary math curriculum, you can take a one-week course in June at the Anja Greer Mathematics, Science and Technology Conference held at Phillips Exeter Academy. For more information go to http://mathconf.exeter.edu/ ❑

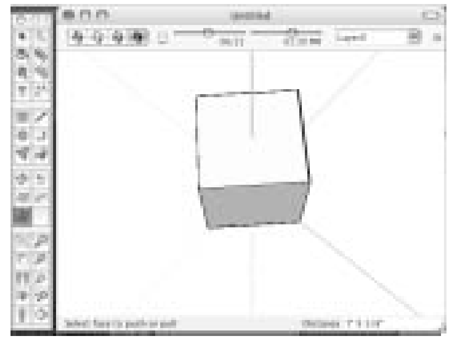
[1] Eric W. Weisstein. "Platonic Solid." From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/PlatonicSolid.html.

[2] If you are interested in why polygons and polyhedra have the names they do go to http://mathforum.org/dr.math/faq/faq.polygon.names.html.

[3] For a good discussion of the Archimedian Solids go to http://www.fact-index.com/a/ar/archimedean_solid.html.



STEP 1.



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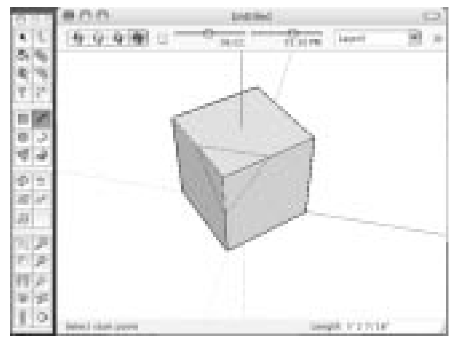

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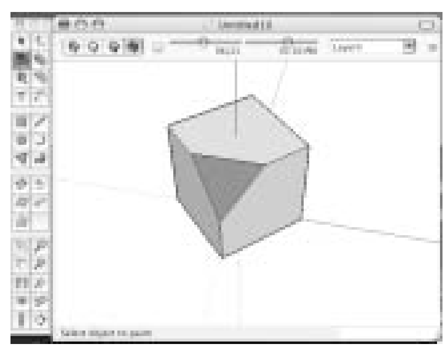

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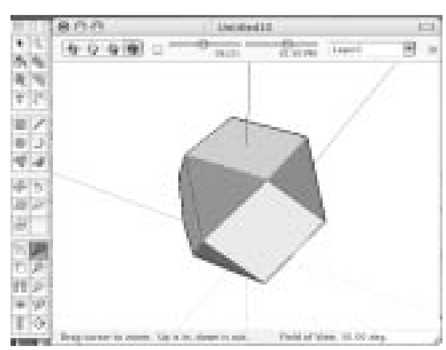

STEP 5.