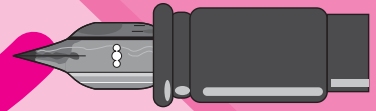


CURVES AHEAD



INTRODUCTION BY JONATHAN CHOATE

The following article was written by several students in my computer graphics class. I have added some diagrams and the Mathematica material but the work was mostly done by them. My next column will deal with homogeneous coordinates and will provide some interesting material for those of you who are using computer algebra systems that have dot products and cross products built in. Also, those of you with historical interests may be find the material interesting because some of the tricks used by graphics programmers today to do things quickly were developed centuries ago when mathematicians were in the process of developing the field of projective geometry.

SHANTANU DHAKA,
REBECCA HUDSON,
BETSY LEE, AND DEREK LEE

Below is a description of how the pen tool works on many of the graphic design programs such as Adobe Illustrator and Macromedia FreeHand 10. The curves created are *Bezier curves* and were named after Pierre Bezier, a French engineer who worked for Renault.

Click anywhere on the page and this will be the point P_0 . While still clicking drag the mouse until it falls where you want P_1 . The image will look like that shown in Figure 1:

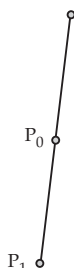


FIGURE 1. FIRST STEP IN CREATING A BEZIER CURVE USING A PEN TOOL.

To get P_2 , immediately click on the place you want it and then drag in the opposite direction of P_2 until the end of the line, the one opposite of where the mouse is, is located where you want P_3 . The Bezier curve is immediately drawn and the image then looks like that in Figure 2.

By clicking on the white arrow and then anywhere in the background only the Bezier curve remains as shown in Figure 3.

There are ways to change the curve by simply moving the lines or stretching points. The simplest way to change the appearance of the curve is to rotate one of the points, P_1 or P_2 . If you rotate P_2 in the counterclockwise direction the image looks like that shown in Figure 4:

In order to see what is going on mathematically with the pen tool, do the following:

Take any four points in the plane P_0, P_1, P_2 , and P_3 as shown in Figure 5 and pick a parameter t such that $0 \leq t \leq 1$ and locate three new points $Q_0(t)$, $Q_1(t)$, and $Q_2(t)$, where

$$Q_0(t) = (1-t)P_0 + tP_1$$

$$Q_1(t) = (1-t)P_1 + tP_2$$

$$Q_2(t) = (1-t)P_2 + tP_3$$

Note that point Q_0 lies on line segment P_0P_1 , point Q_1 lies on line segment P_1P_2 and point Q_2 lies on line segment P_2P_3 ,

Now locate two new points $R_0(t)$ and $R_1(t)$ where

$$R_0(t) = (1-t)Q_0(t) + tQ_1(t),$$

$$R_1(t) = (1-t)Q_1(t) + tQ_2(t),$$

Next locate one final point S_0 where

$$S_0(t) = (1-t)R_0(t) + tR_1(t),$$

Using substitution and a lot of algebraic simplification algebra, the point $S_0(t)$ can be expressed in terms of the original four points P_0, P_1, P_2 , and P_3 .

$$S_0 = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3$$

You now have a point S_0 which is defined in terms of a parameter t and the four points P_0, P_1, P_2 and P_3 . If you vary t from 0 to 1 and plot the resulting points you will generate a curve. In fact, if you chose the four points to correspond to the four points in Figure 5, you would get a curve which very closely approximated the curve generated by Illustrator as shown

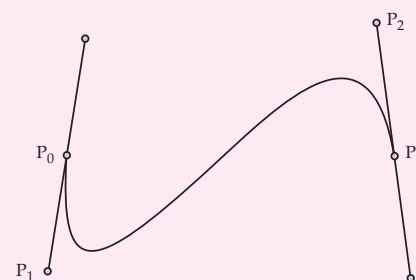


FIGURE 2. SECOND STEP IN CREATING A BEZIER CURVE USING A PEN TOOL.

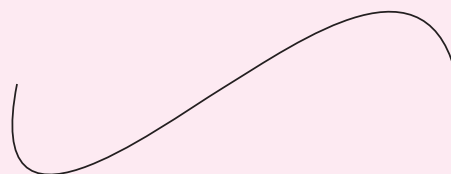


FIGURE 3. THIRD STEP IN CREATING A BEZIER CURVE USING A PEN TOOL.

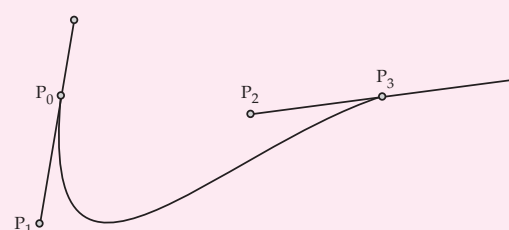


FIGURE 4. MODIFYING THE ORIGINAL CURVE BY MOVING A CONTROL POINT.

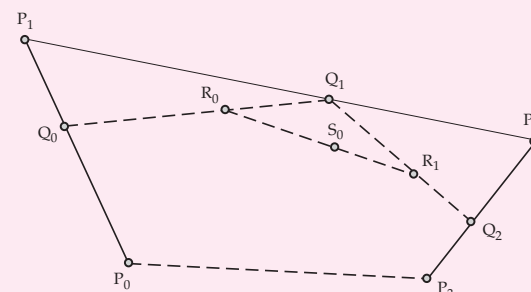


FIGURE 5. CONSTRUCTING A BEZIER CURVE

in **Figure 6**. This curve is called a Bezier curve and the algorithm used to construct it was developed by Paul de Casteljou while he was working for Citroën. The points P_0 and P_3 are called anchor points and P_1 and P_2 are called handles or control points. By moving P_0 and P_3 you change where the curve begins and ends and by moving P_1 and P_2 you change the shape of the curve. Neat!

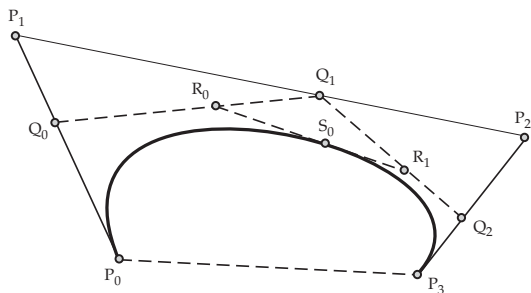


FIGURE 6. A BEZIER CURVE CREATED WITH GEOMETERS SKETCHPAD USING THE DE CASTELJOU ALGORITHM.

There are lots of ways you can generate Bezier curves. If you have access to a computer algebra system, you can use it to convert the point equation for S_0 into a set of parametric equations and then plot them using a graphing program or a graphing calculator (see Figure 7).

If $S_0 = (x(t), y(t))$ then a little algebra yields:

$$x(t) = (-x_0 + 3x_1 - 3x_2 + x_3)t^3 + (3x_0 - 6x_1 + 3x_2)t^2 + (-3x_0 + 3x_1)t + x_0$$

$$y(t) = (-y_0 + 3y_1 - 3y_2 + y_3)t^3 + (3y_0 - 6y_1 + 3y_2)t^2 + (-3y_0 + 3y_1)t + y_0$$

Bezier curves can be studied at lots of different levels. Here are some things to think about:

1. The points P_0, P_1, P_2 and P_3 define a quadrilateral. What would the resulting Bezier curve look like if $P_0P_1P_2P_3$ was a
 - convex quadrilateral
 - concave quadrilateral
 - a square
 - a quadrilateral with intersecting sides

2. If P_1 and P_2 lie on segment P_0P_3 , then the quadrilateral becomes a line segment. How is the curve plotted if the four points lie in the following order?
 - i) $P_0P_1P_3P_2$
 - ii) $P_0P_2P_1P_3$
 - iii) $P_1P_0P_2P_3$

3. If you know some calculus try these:

- What kind of quadrilateral would $P_0P_1P_2P_3$ have to be in order for $\frac{dy}{dx}$ to be undefined and what would the resulting Bezier curve look like?
- What kind of quadrilateral would $P_0P_1P_2P_3$ have to be in order for the Bezier curve to have a point of inflection?
- Bezier curves tend to be far more curvy than the curves studied in a traditional calculus course. They provide a wonderful way to introduce the concept of curvature, a measure of how a curve bends as you move along it. The curvature of a curve at a point P is defined to be the rate of change of the angle

formed by the tangent line to the curve at P and the positive x -axis as a function of the arc length S from some arbitrary point on the curve to P . For a good discussion of the derivation of a formula $\frac{d\theta}{ds}$ go to <http://mathworld.wolfram.com/curvature.html>

If you are interested in learning more about these fascinating curves, we highly recommend that you go to the two Websites given below. The first develops more of the mathematics behind the curves and has some interesting applets. The second discusses the curves from a designer's point of view and shows how they are used to design typefaces and airfoils. □

<http://www.moshplant.com/direct-or/bezier/index.html>

<http://graphicssoft.about.com/library/tuts/bltut16.html>

*Send solutions to old problems and any new ideas to the Modeler's Corner editor:
Jonathan Choate, Groton School,
Box 991, Groton, MA 01450.*

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In[1]:=P0={x0, y0}
Out[1]={x0, y0}
In[2]:=P1={x1, y1}
Out[2]={x1, y1}
In[3]:=P2={x2, y2}
Out[3]={x2, y2}
In[4]:=P3={x3, y3}
Out[4]={x3, y3}
In[5]:=S0=(1-t)^3P0+3(1-t)^2tP1+3(1-t)t^2P2+t^3P3
Out[5]={x0(1-t)^3+3x1(1-t)^2t+3x2t^2(1-t)+t^3x3,
        y0(1-t)^3+3y1(1-t)^2t+3y2t^2(1-t)+t^3y3}
In[6]:=Expand[S0]
Out[6]={-x0t^3+3x1t^2-3x2t+3x3,
        -y0t^3+3y1t^2-3y2t+3y3}
```

FIGURE 7. THE MATHEMATICA DERIVATION OF THE PARAMETRIC EQUATION FOR A BEZIER CURVE.